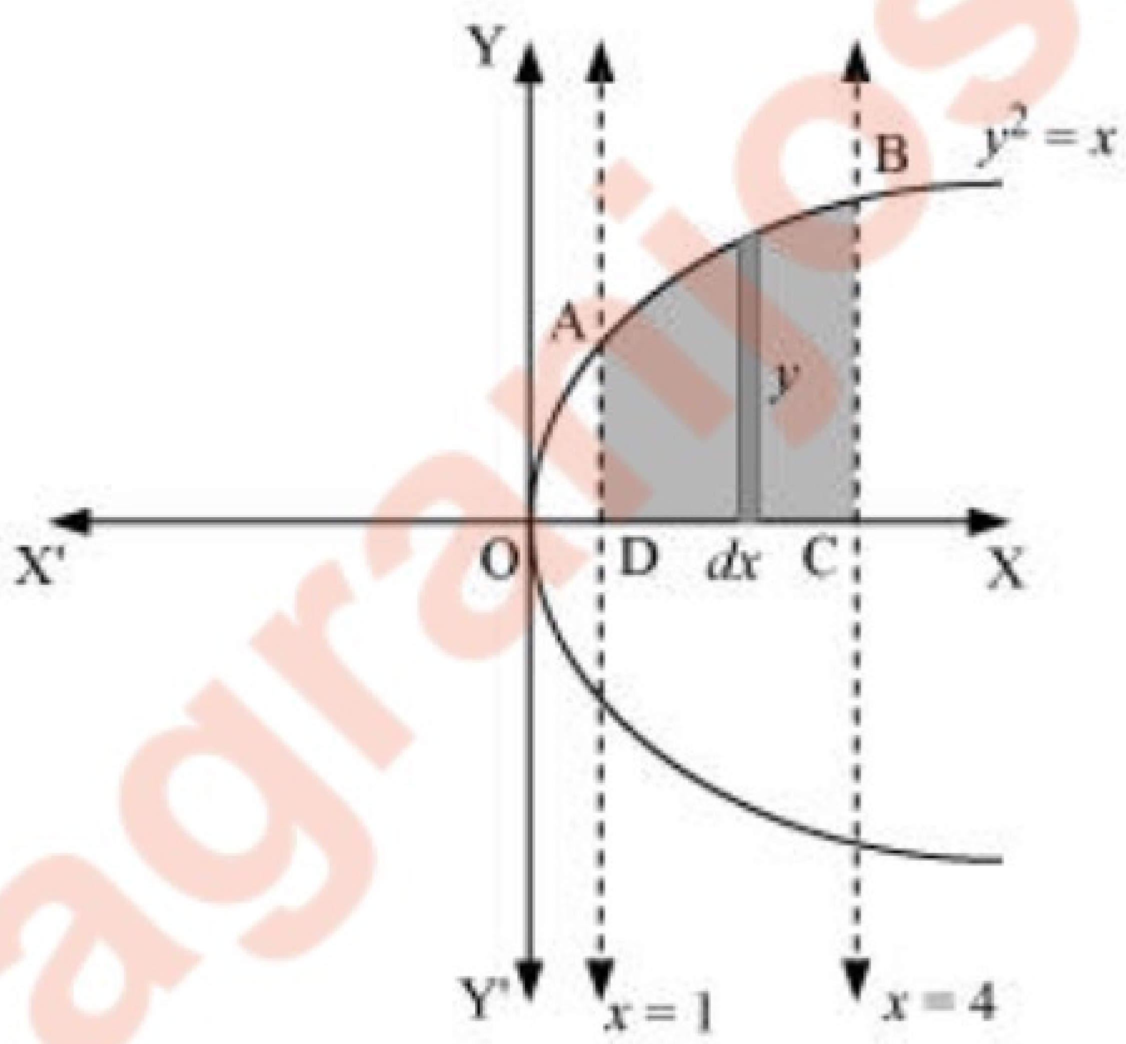
Chapter - 8
Application of Integrals
Class – XII
Subject –Maths

Exercise- 8.1

1. **Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis.**

Sol.



Area of region ABCD is

$$= \int_1^4 y\,dx$$

$$= \int_1^4 \sqrt{x}\,dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3}\right]_1^4$$

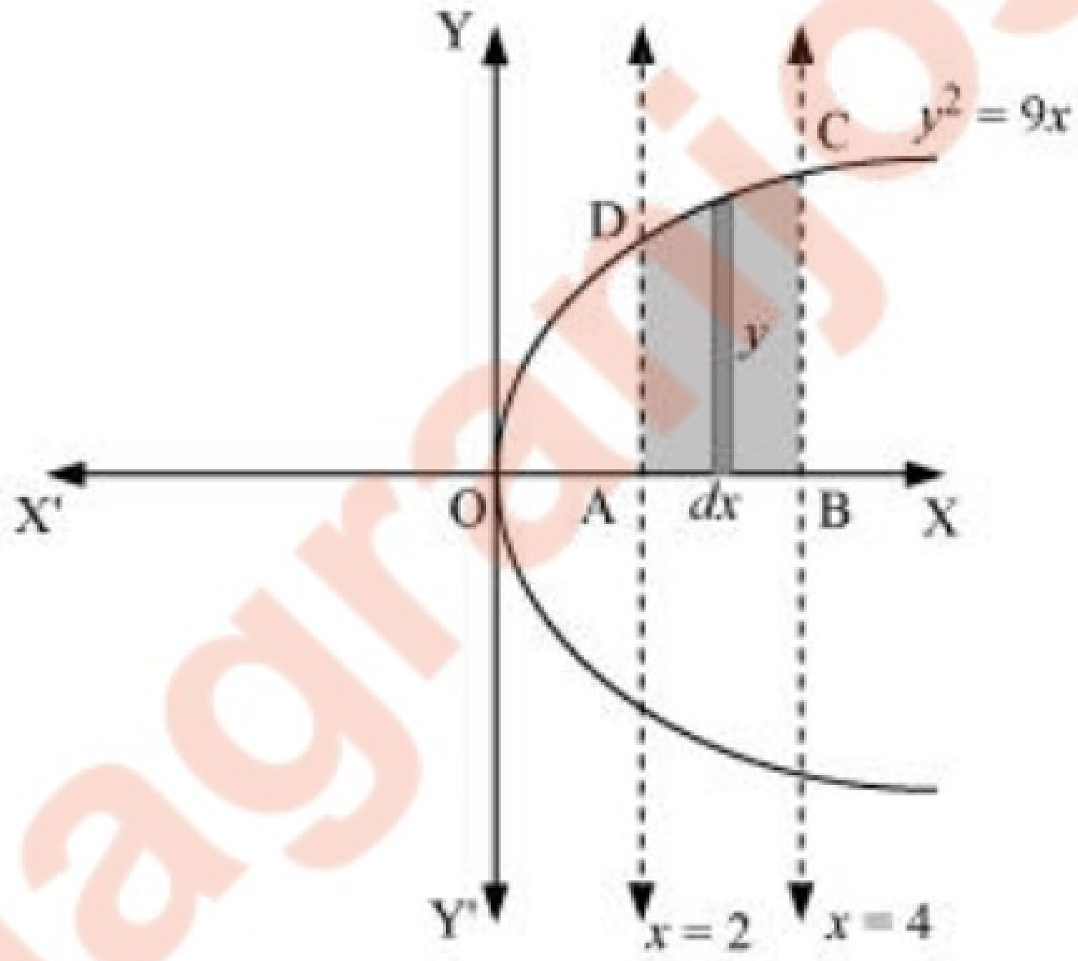
$= \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$

$= \frac{2}{3}(8-1)$

$= \frac{14}{3}$ units

2. **Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.**

Sol.



Area of region ABCD is

$= \int_2^4 y\,dx$

$= \int_2^4 3\sqrt{x}\,dx$

$= 3\left[\frac{2x^{\frac{3}{2}}}{3}\right]_2^4$

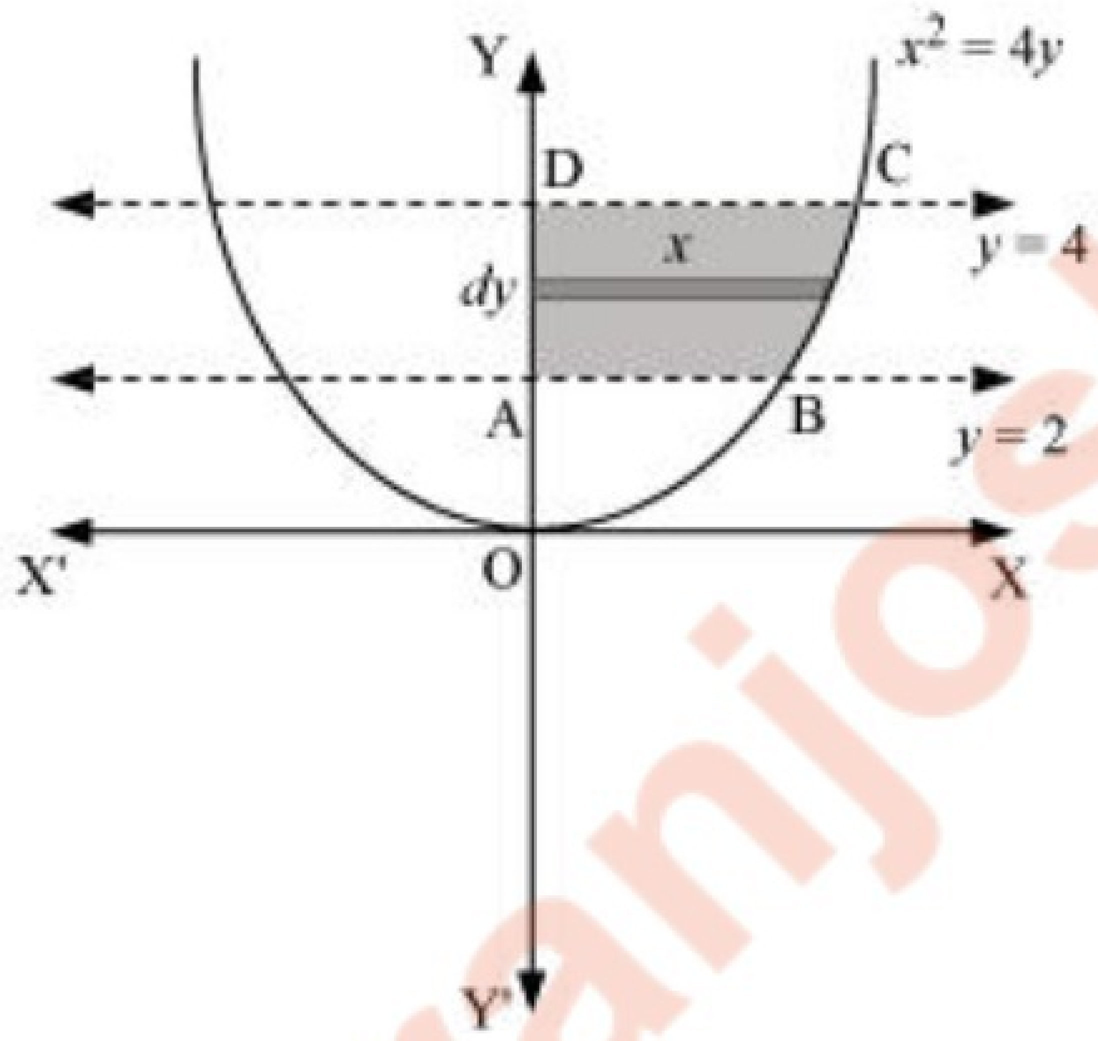
$= 2\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$

$= 2\left(8 - 2\sqrt{2}\right)$

$= \left(16 - 4\sqrt{2}\right)$ units

3. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

Sol.



Area of region ABCD is

$= \int_2^4 x\,dy$

$= \int_2^4 2\sqrt{y}\,dy$

$= 2\left[\frac{2y^{\frac{3}{2}}}{3}\right]_2^4$

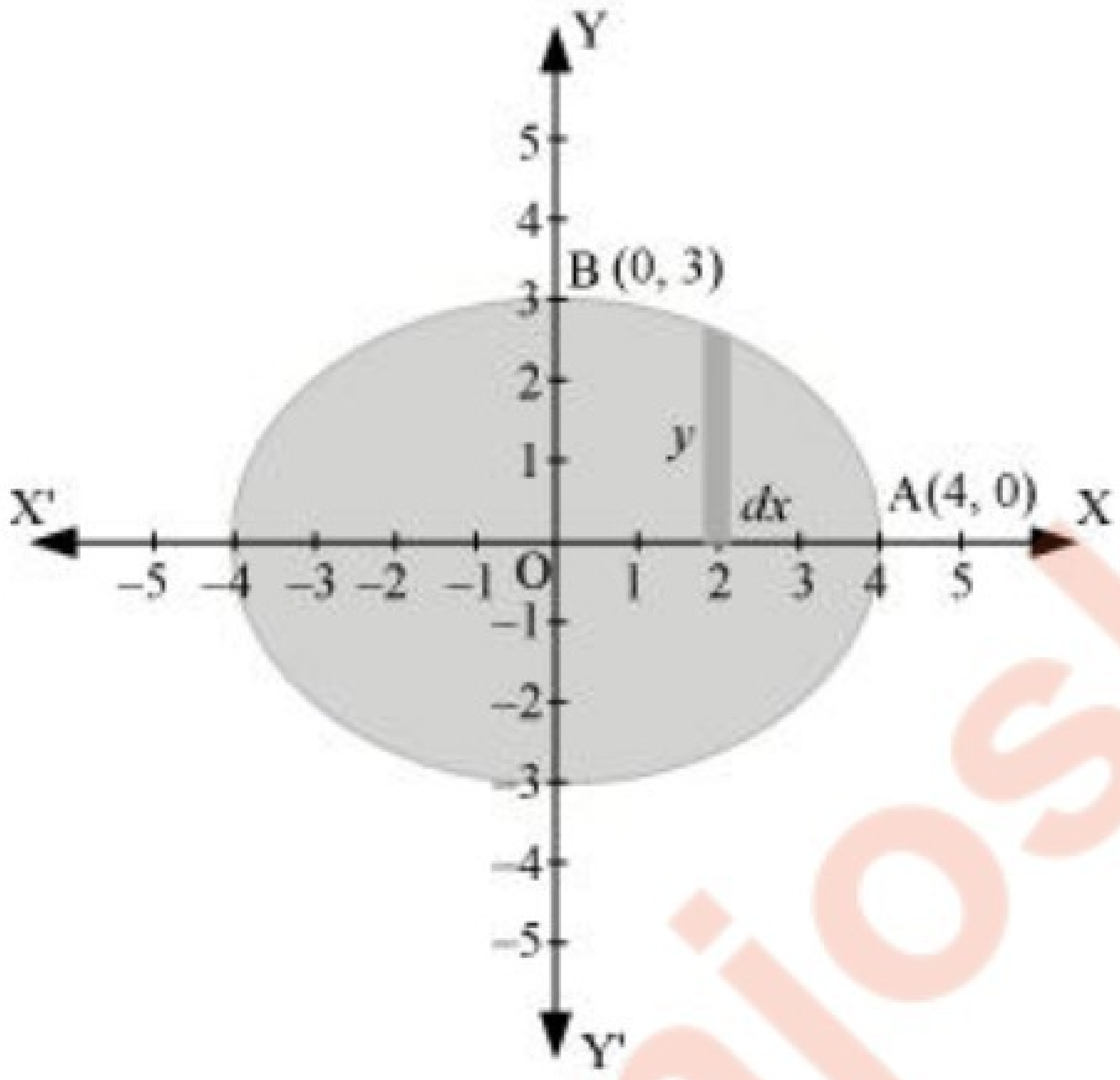
$= \frac{4}{3}\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$

$= \frac{4}{3}\left(8 - 2\sqrt{2}\right)$

$= \left(\frac{32 - 8\sqrt{2}}{3}\right)$ units

4. **Find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$**

Sol.



Area bounded by ellipse = 4 × Area of OAB

Now, Area of OAB is

$$=\int_0^4 y\,dx$$

$$=\int_0^4 3\sqrt{\left(1-\frac{x^2}{16}\right)}dx$$

$$=\frac{3}{4}\int_0^4\sqrt{\left(16-x^2\right)}dx$$

$$=\frac{3}{4}\left[\frac{x}{2}\sqrt{\left(16-x^2\right)}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= \frac{3}{4}\left[\frac{4}{2}\sqrt{\left(16-4^2\right)} + \frac{16}{2}\sin^{-1}\frac{4}{4} - \frac{16}{2}\sin^{-1}\left(0\right)\right]$$

$$= \frac{3}{4}\left[8\sin^{-1}\left(1\right) - 8\sin^{-1}\left(0\right)\right]$$

$$= \frac{3}{4}\left[8.\frac{\pi}{2} - 0\right]$$

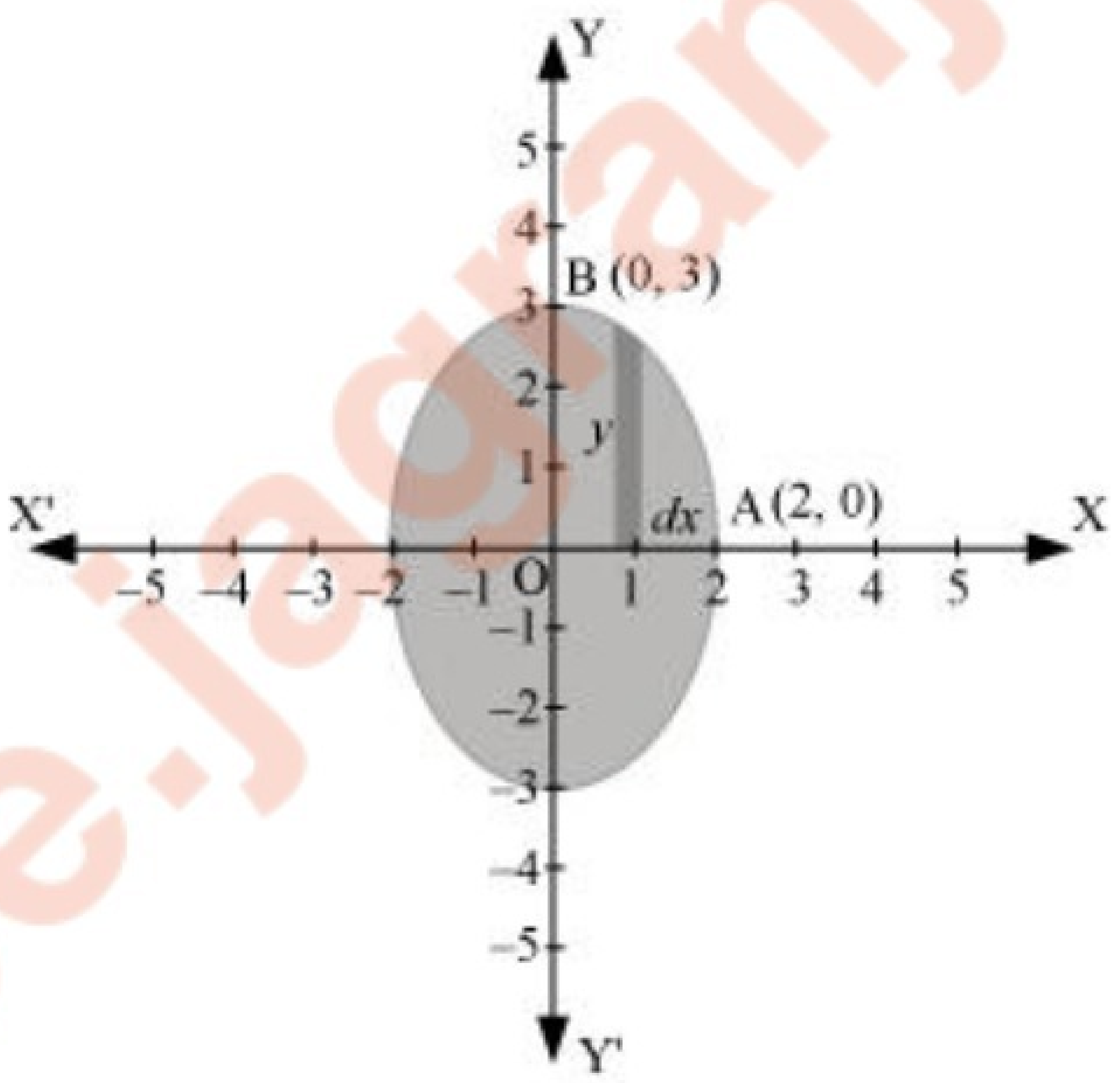
$$= \frac{3}{4}\left(4\pi\right)$$

$$= 3\pi$$

Therefore, area bounded by the ellipse = $4 \times 3\pi = 12\pi$ units

5. Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Sol.



Equation of ellipse is:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$y = 3\sqrt{1 - \frac{x^2}{4}}$$

Thus,

Area bounded by ellipse = 4 × Area OAB

Area of region OAB is

$$= \int_0^2 y\,dx$$

$$= \int_0^2 3\sqrt{\left(1 - \frac{x^2}{4}\right)}dx$$

$$= \frac{3}{2}\int_0^2 \sqrt{\left(4 - x^2\right)}dx$$

$$= \frac{3}{2}\left[\frac{x}{2}\sqrt{\left(4 - x^2\right)} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= \frac{3}{2}\left[\frac{2}{2}\sqrt{\left(4 - 2^2\right)} + \frac{4}{2}\sin^{-1}\frac{2}{2} - \frac{4}{2}\sin^{-1}(0)\right]$$

$$= \frac{3}{2}\left[2\sin^{-1}(1)\right]$$

$$= \frac{3}{2}\left[2.\frac{\pi}{2}\right]$$

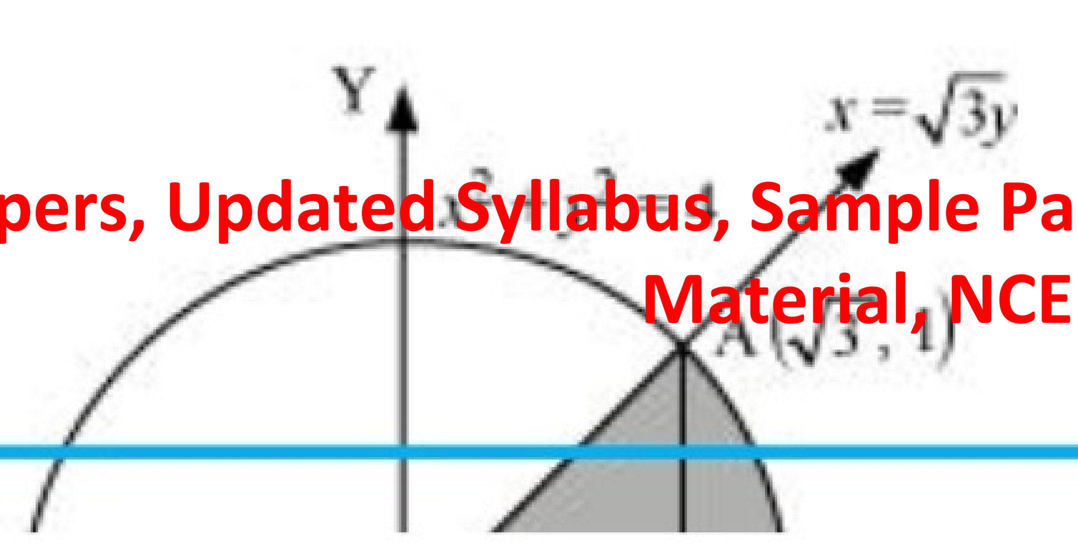
$$= \frac{3}{2}(\pi)$$

$$= \frac{3\pi}{2}$$

Hence, area bounded by the ellipse = $4 \times \frac{3\pi}{2} = 6\pi$ units

6. Find the area of the region in the first quadrant enclosed by *x*-axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$

Sol.



Area OAB = Area ΔOCA + Area ACB

Area of OAC $=\frac{1}{2}\times OC\times AC=\frac{1}{2}\times\sqrt{3}\times 1=\frac{\sqrt{3}}{2}$

Area of ABC is

$$=\int_{\sqrt{3}}^{2} y\,dx$$

$$=\int_{\sqrt{3}}^{2}\sqrt{\left(4-x^2\right)}dx$$

$$=\left[\frac{x}{2}\sqrt{\left(4-x^2\right)}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$$

$$= \left[\frac{2}{2}\sqrt{\left(4-2^2\right)} + \frac{4}{2}\sin^{-1}\frac{2}{2} - \frac{\sqrt{3}}{2}\sqrt{\left(4-3\right)} - \frac{4}{2}\sin^{-1}\frac{\sqrt{3}}{2}\right]$$

$$= \left[2\sin^{-1}(1) - \frac{\sqrt{3}}{2} - 2\sin^{-1}\frac{\sqrt{3}}{2}\right]$$

$$= \left[2.\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2.\frac{\pi}{3}\right]$$

$$= \left[\pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3}\right]$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

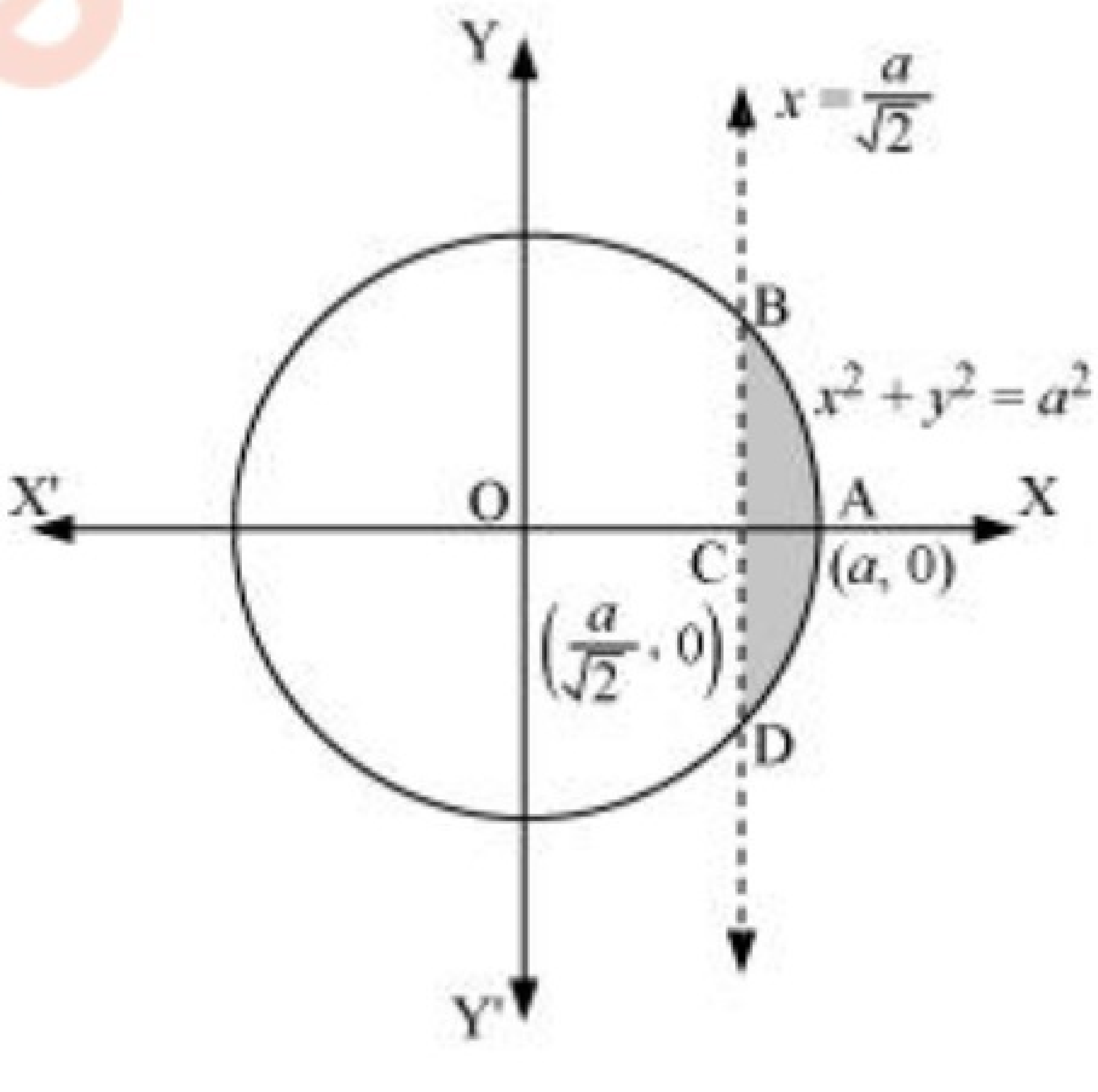
Area OAB = Area ΔOCA + Area ACB

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{3} \textit{ units}$$

7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$

Sol.



We can observe that the area ABCD is symmetrical about x-axis.

∴ Area ABCD = 2 × Area ABC

Area ABC is

$$= \int_{\frac{a}{\sqrt{2}}}^{a} y\,dx$$

$$= \int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{\left(a^2 - x^2\right)}\,dx$$

$$= \left[\frac{x}{2}\sqrt{\left(a^2 - x^2\right)} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$$

$$= \left[\frac{a}{2}\sqrt{\left(a^2 - a^2\right)} + \frac{a^2}{2}\sin^{-1}\frac{a}{a} - \frac{a}{2\sqrt{2}}\sqrt{\left(a^2 - \frac{a^2}{2}\right)} - \frac{a^2}{2}\sin^{-1}\frac{a}{\sqrt{2}a}\right]$$

$$= \left[\frac{a^2}{2}\sin^{-1}(1) - \frac{a}{2\sqrt{2}}.\frac{a}{\sqrt{2}} - \frac{a^2}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= \left[\frac{a^2}{2}.\frac{\pi}{2} - \frac{a^2}{4} - \frac{a^2}{2}.\frac{\pi}{4}\right]$$

$$= \left[\frac{a^2}{2}.\frac{\pi}{4} - \frac{a^2}{4}\right]$$

$$= \frac{a^2}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{a^2}{4}\left[\frac{\pi}{2} - 1\right]$$

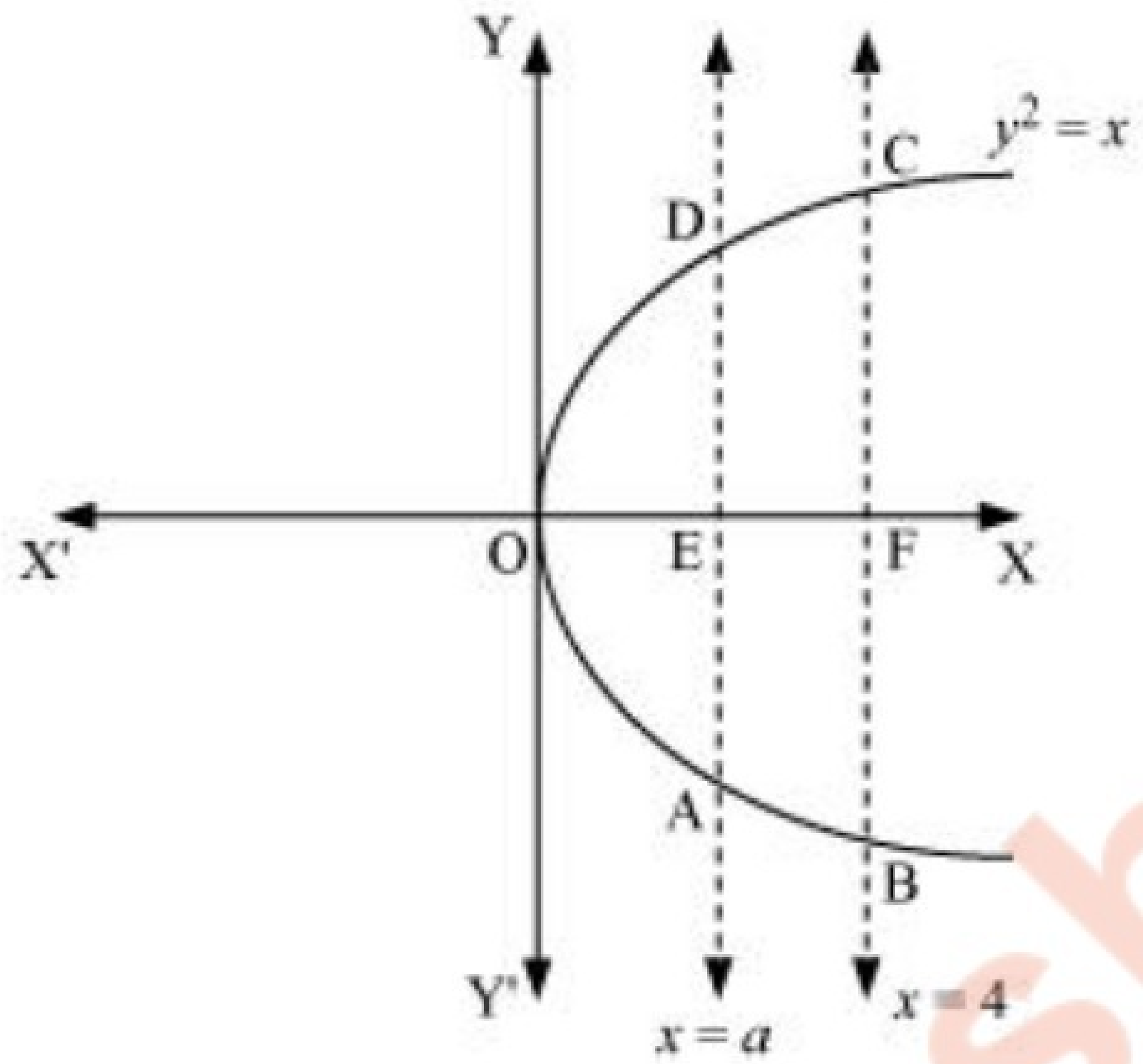
Area ABCD = 2 × Area ABC

$$= 2.\frac{a^2}{4}\left[\frac{\pi}{2} - 1\right]$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} - 1\right]$$

8. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

Sol.



The line, $x = a$, divides the area bounded by the parabola and $x = 4$ into two equal parts.

∴ Area OAD = Area ABCD

We can observe that the given area is symmetrical about x-axis.

Thus, Area OED = Area EFCD

Area of region OED is

$$= \int_0^a y\,dx$$

$$= \int_0^a \sqrt{x}\,dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^a$$

$$= \frac{2}{3}(a)^{\frac{3}{2}}$$

Area of region EFCD is

$$= \int_a^4 \sqrt{x}\,dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3}\right]_a^4$$

$$= \frac{2}{3}\left(8 - a^{\frac{3}{2}}\right)$$

Now,

Area OED = Area EFCD

$$\frac{2}{3}(a)^{\frac{3}{2}} = \frac{2}{3}\left(8 - a^{\frac{3}{2}}\right)$$

$$(a)^{\frac{3}{2}} = \left(8 - a^{\frac{3}{2}}\right)$$

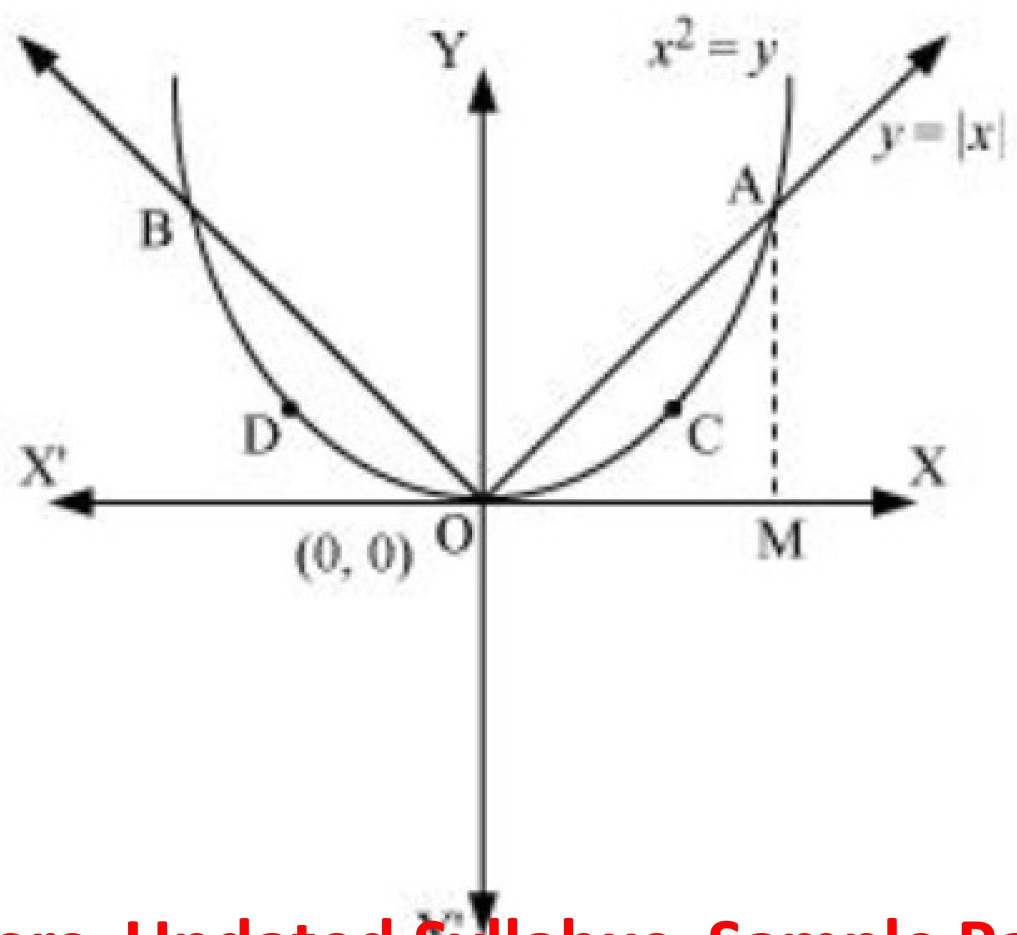
$$2(a)^{\frac{3}{2}} = 8$$

$$(a)^{\frac{3}{2}} = 4$$

$$a = (4)^{\frac{2}{3}}$$

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$

Sol.



The given area is symmetrical about y-axis.

Thus,

Area OACO = Area ODBO

Area of OACO = Area ΔOAB – Area OBACO

Area ΔOAB = $\frac{1}{2} \times OB \times AB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

Area of region OBACO is

$$= \int_0^1 y dx$$

$$= \int_0^1 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

Now,

Area of OACO = Area of ΔOAB – Area of OBACO

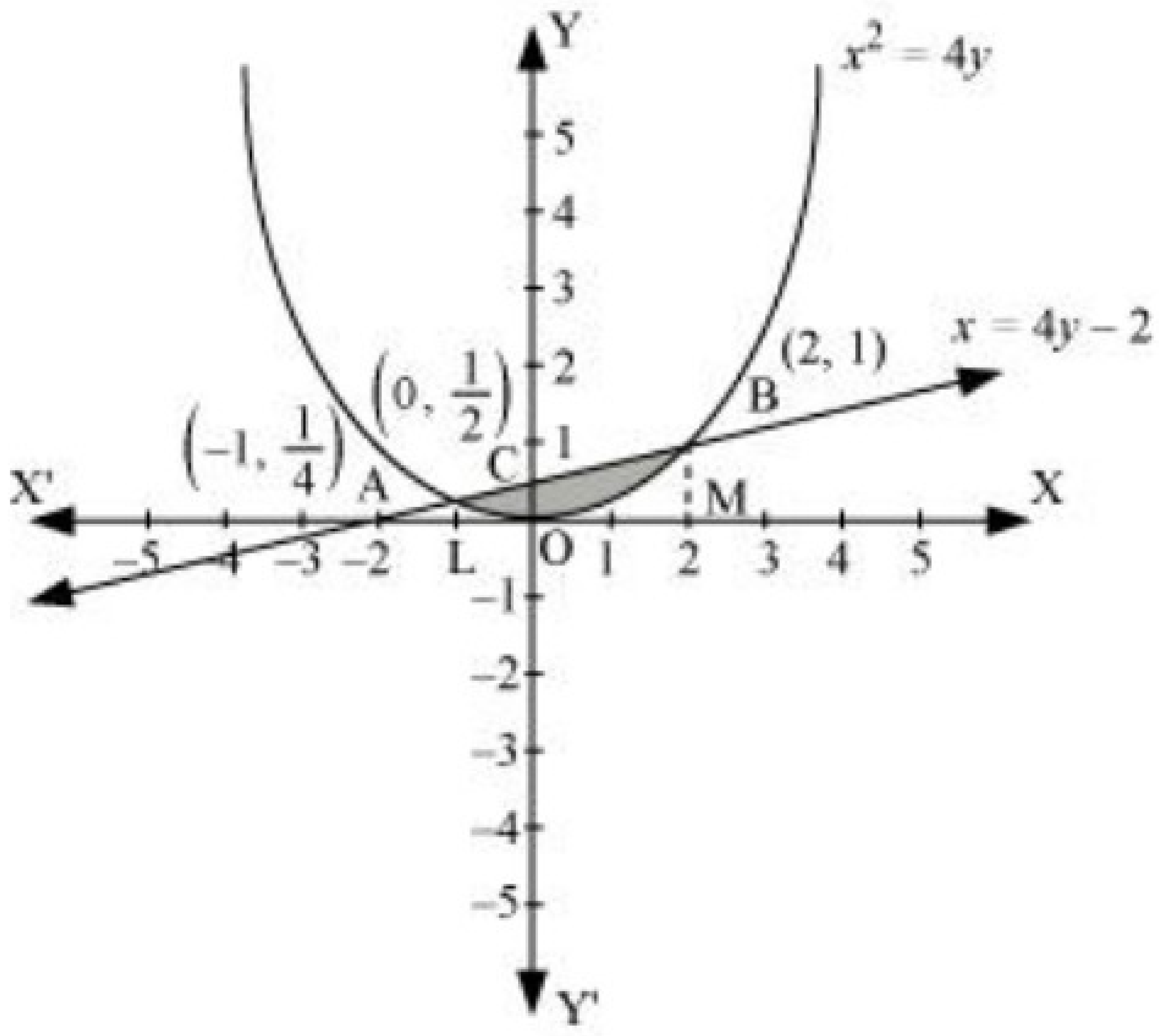
$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

Therefore, required area = $= 2.\frac{1}{6} = \frac{1}{3}$ units

10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$

Sol.



From the figure,

Area OBAO = Area OBCO + Area OACO

Thus,

Area OBCO = Area OMBC – Area OMBO

$$= \int_0^2 \frac{x+2}{4} dx - \int_0^2 \frac{x^2}{4} dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{4}(2+4) - \frac{1}{4}.\frac{8}{3}$$

$$= \frac{3}{2} - \frac{2}{3}$$

$$= \frac{5}{6}$$

Similarly,

Area OACO = Area OLAC – Area OLAO

$$= \int_{-1}^{0} \frac{x+2}{4} dx - \int_{-1}^{0} \frac{x^2}{4} dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{0} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{0}$$

$$= -\frac{1}{4}\left(\frac{1}{2} - 2\right) - \left[-\frac{1}{4}.\left(\frac{-1}{3}\right)\right]$$

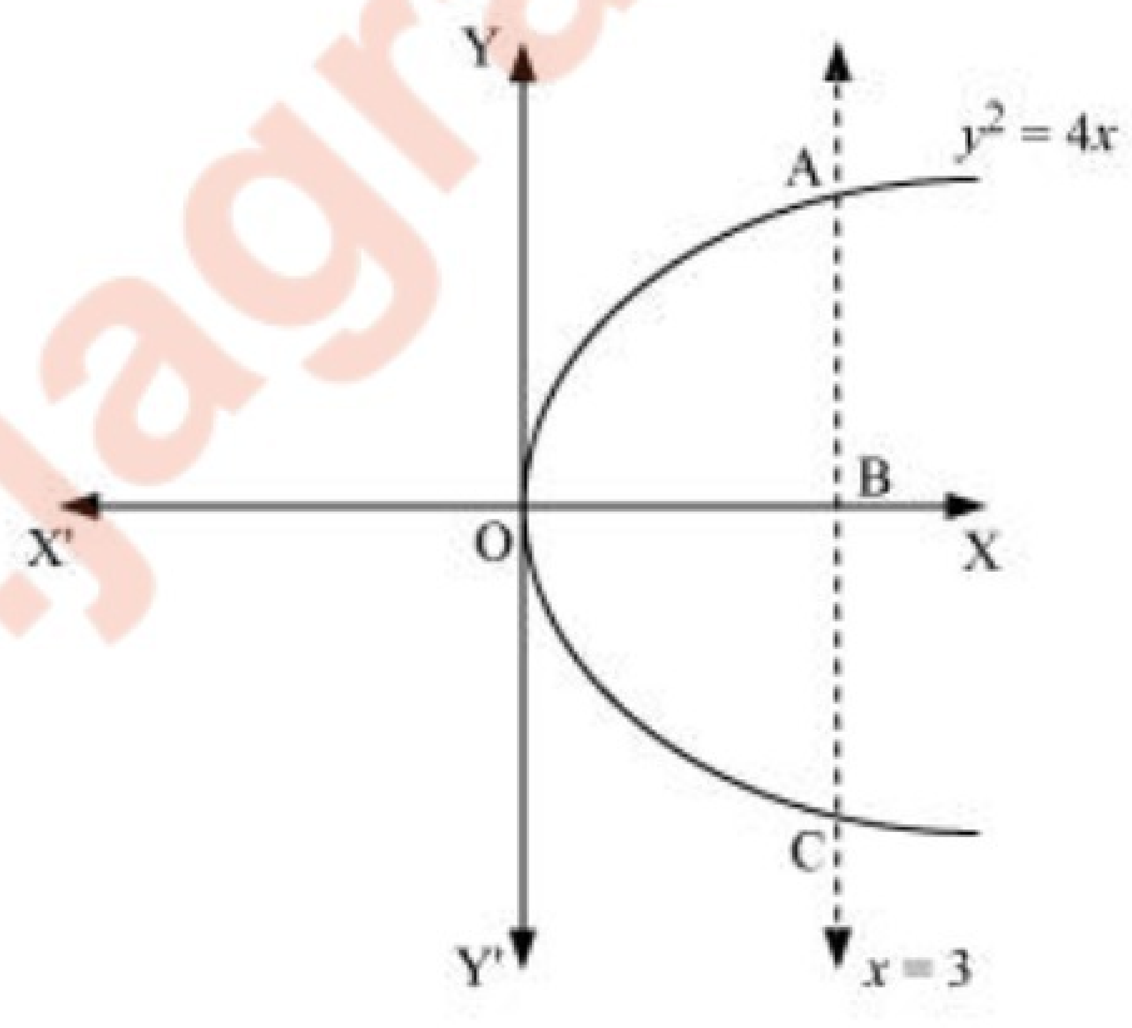
$$= \frac{3}{8} - \frac{1}{12}$$

$$= \frac{7}{24}$$

Hence, required area $= \frac{7}{24} + \frac{5}{6} = \frac{9}{8}$ units

11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$

Sol.



The area OACO is symmetrical about x-axis.

Thus,

Area of OACO = 2 (Area of OAB)

Area of region OACO is

$$= 2\left[\int_0^3 y dx\right]$$

$$= 2\int_0^3 2\sqrt{x} dx$$

$$= 4.\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^3$$

$$= \frac{8}{3}(3)^{\frac{3}{2}}$$

$$= 8\sqrt{3} \text{ units}$$

12. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

A. π

B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{4}$

Sol.

Area OAB is

$$= \int_0^2 y\,dx$$

$$= \int_0^2 \sqrt{4 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + 4\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2.\frac{\pi}{2}$$

$$= \pi \text{ units}$$

Hence, the correct answer is A.

13. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

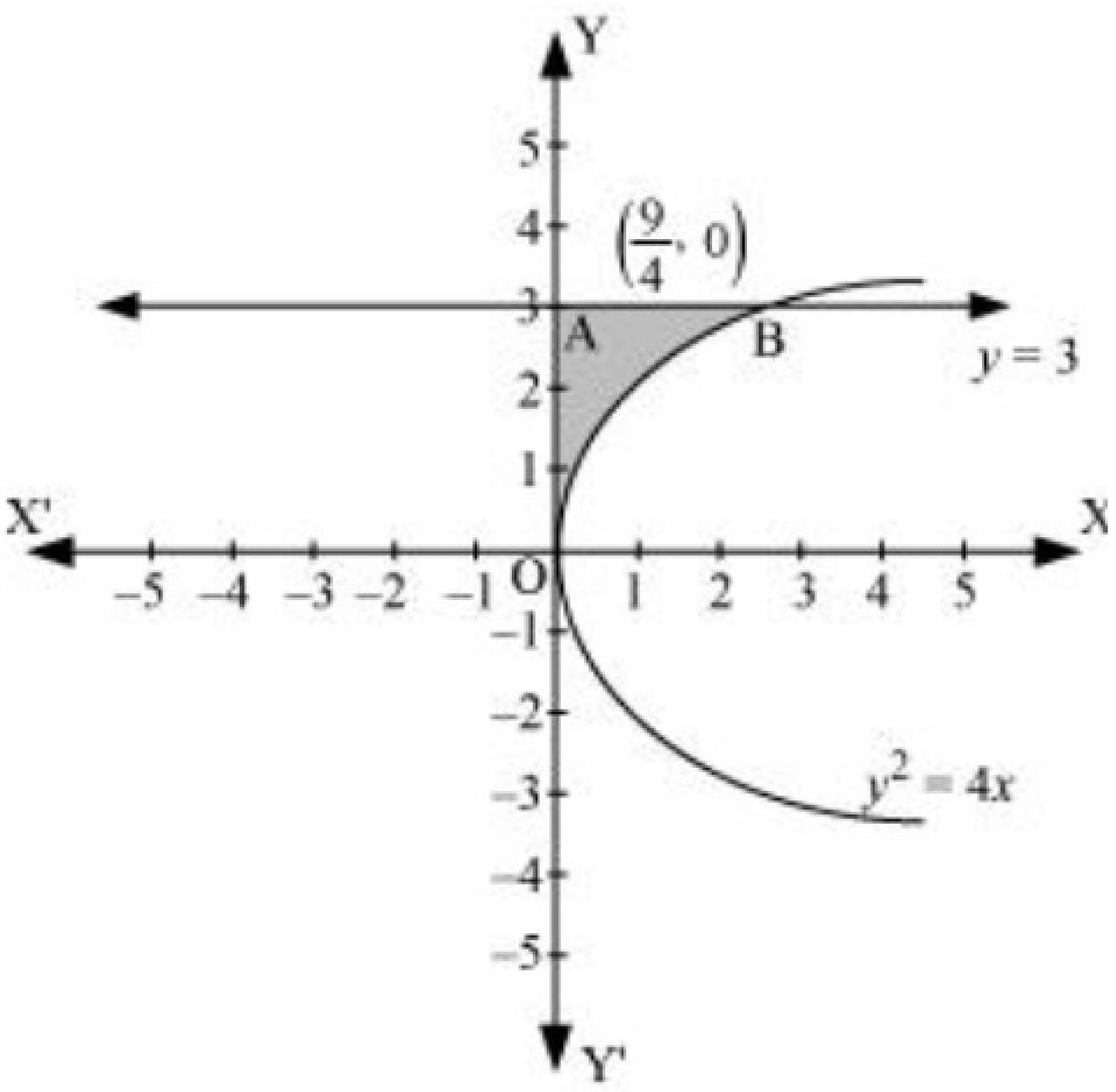
A. 2

B. $\frac{9}{4}$

C. $\frac{9}{3}$

D. $\frac{9}{2}$

Sol.



Area OAB is

$$= \int_0^3 x\,dy$$

$$= \int_0^3 \frac{y^2}{4}\,dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$$

$$= \frac{1}{12}(27)$$

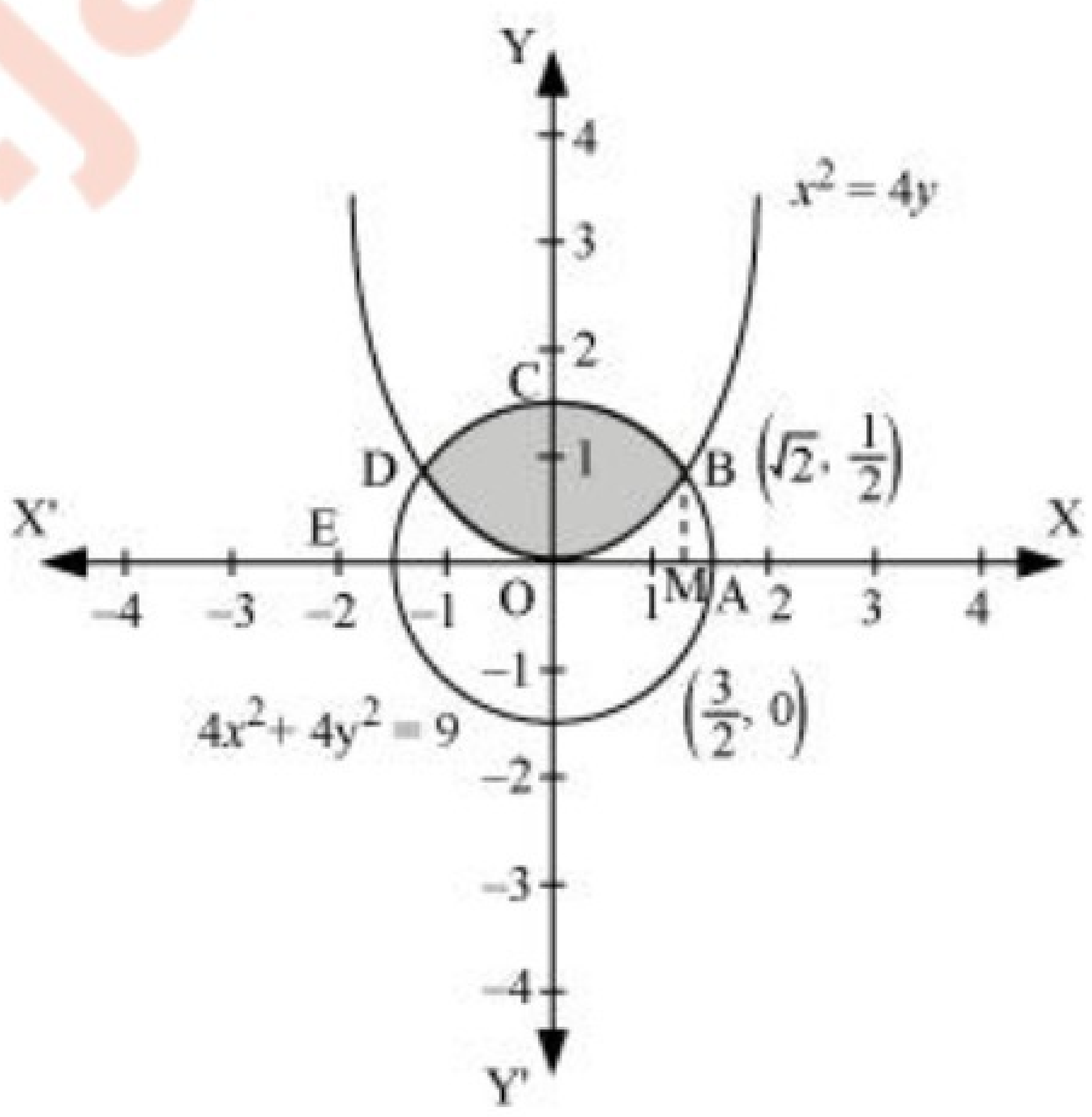
$$= \frac{9}{4}\ units$$

Hence, the correct answer is B.

Exercise-8.2

1. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$

Sol.



The required area is represented by the shaded area OBCDO.

We can observe that the required area is symmetrical about y-axis.

Thus,

Area OBCDO = 2 × Area OBCO

Draw BM perpendicular to OA.

Thus,

Area OBCO = Area OMBCO – Area OMBO

$$= \int_0^{\sqrt{2}} \sqrt{\left(\frac{9-4x^2}{4}\right)}dx - \int_0^{\sqrt{2}} \sqrt{\left(\frac{x^2}{4}\right)}dx$$

$$= \frac{1}{2}\int_0^{\sqrt{2}} \sqrt{\left(9-4x^2\right)}dx - \frac{1}{4}\int_0^{\sqrt{2}} \sqrt{x^2}\,dx$$

$$= \frac{1}{4}\left[x\sqrt{\left(9-4x^2\right)} + \frac{9}{2}\sin^{-1}\frac{2x}{3}\right]_0^{\sqrt{2}} - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}}$$

$$= \frac{1}{4}\left[\sqrt{2}\sqrt{(9-8)} + \frac{9}{2}\sin^{-1}\frac{2\sqrt{2}}{3}\right] - \frac{1}{12}.\left(\sqrt{2}\right)^3$$

$$= \frac{1}{4}\left[\sqrt{2} + \frac{9}{2}\sin^{-1}\frac{2\sqrt{2}}{3}\right] - \frac{1}{12}.2\sqrt{2}$$

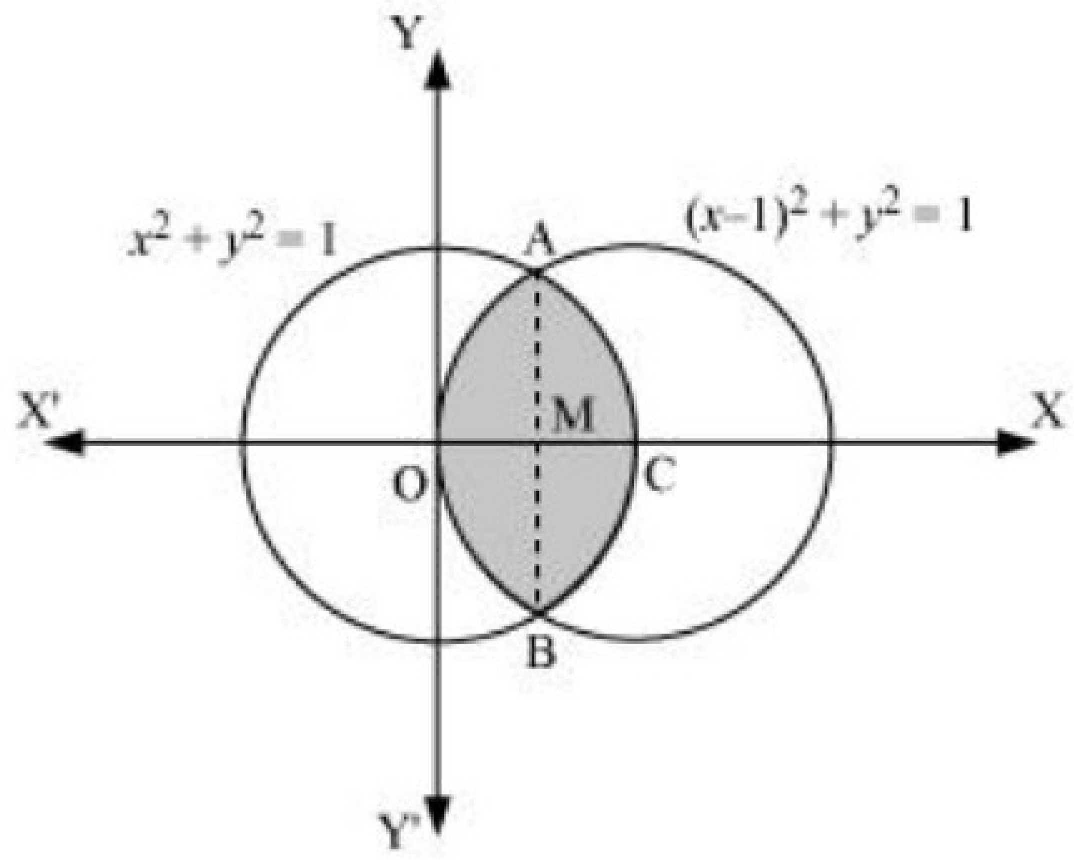
$$= \left[\frac{1}{2\sqrt{2}} + \frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}\right] - \frac{1}{6}.\sqrt{2}$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{3} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right]$$

2. Find the area bounded by curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$

Sol.



The area bounded by the curves, is represented by the shaded area.

We can observe that the required area is symmetrical about x-axis.

Thus,

Area OBCAO = 2 × Area OCAO

Construction: - join AB, which intersects OC at M, such that $AM \perp OC$.

Now,

Area OCAO = Area OMAO + Area MCAM

$$= \int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\,dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,dx$$

$$= \left[\frac{x-1}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}} + \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$= \frac{-1}{4}\sqrt{1-\left(-\frac{1}{2}\right)^2} + \frac{1}{2}\sin^{-1}\left(\frac{1}{2}-1\right) - \frac{1}{2}\sin^{-1}(-1) + \frac{1}{2}\sin^{-1}1 - \frac{1}{4}\sqrt{1-\frac{1}{4}} - \frac{1}{2}\sin^{-1}\frac{1}{2}$$

$$= \frac{-1}{2}\sqrt{\frac{3}{4}} + \frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right) - \frac{1}{2}\sin^{-1}(-1) + \frac{1}{2}\sin^{-1}1 - \frac{1}{2}\sin^{-1}\frac{1}{2}$$

$$= \frac{-\sqrt{3}}{4} + \frac{1}{2}\left(-\frac{\pi}{6}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(\frac{\pi}{6}\right)$$

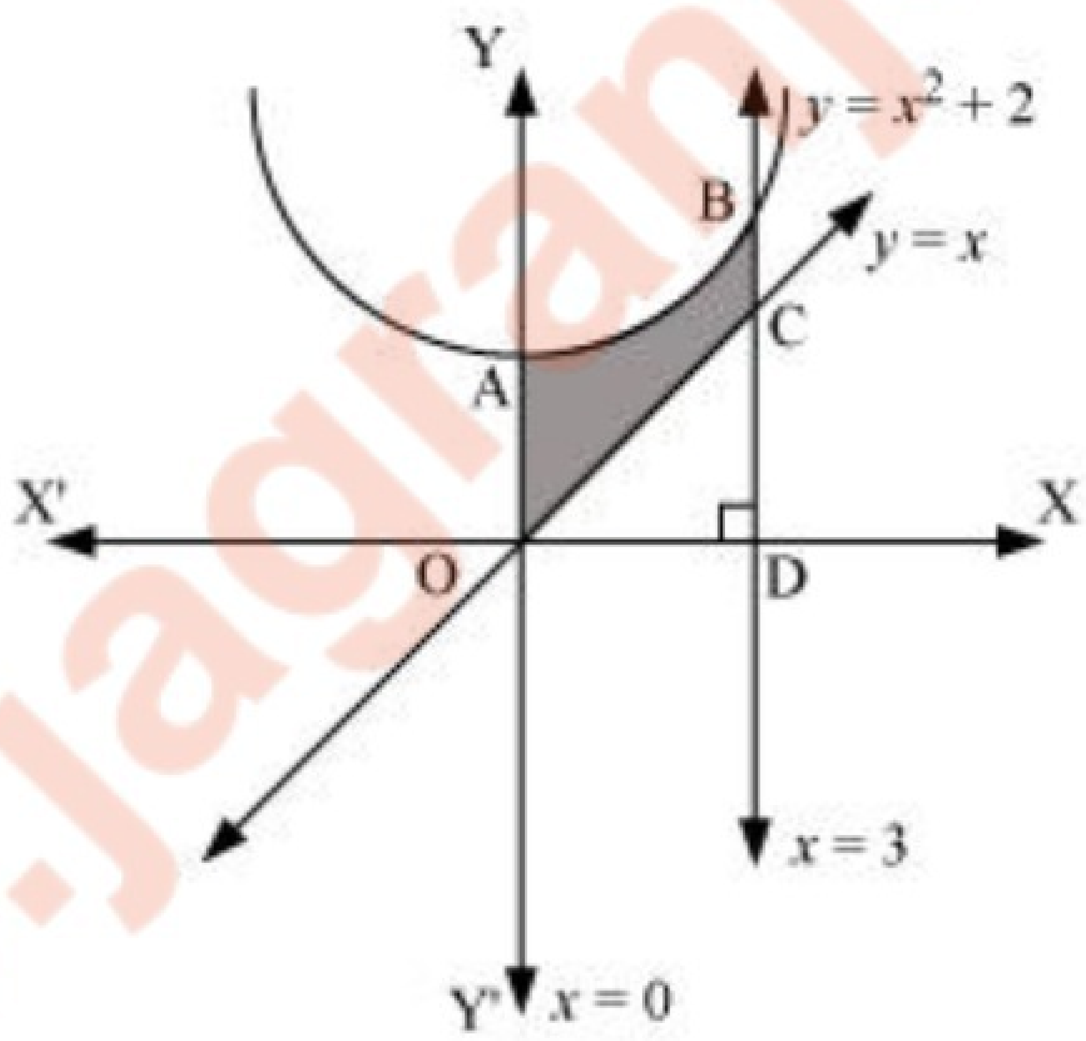
$$= \frac{-\sqrt{3}}{4} - \frac{\pi}{12} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{12}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

Therefore, required area OBCAO $= 2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = \left[\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right]$ units

3. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$

Sol.



The area bounded by the curves & lines is represented by the shaded area OCBAO.

Then, Area OCBAO = Area ODBAO – Area ODCO

$$= \int_0^3 (x^2 + 2)dx - \int_0^3 xdx$$

$$= \left[\frac{x^3}{3} + 2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3$$

$$= (9 + 6) - \frac{9}{2}$$

$$= 15 - \frac{9}{2}$$

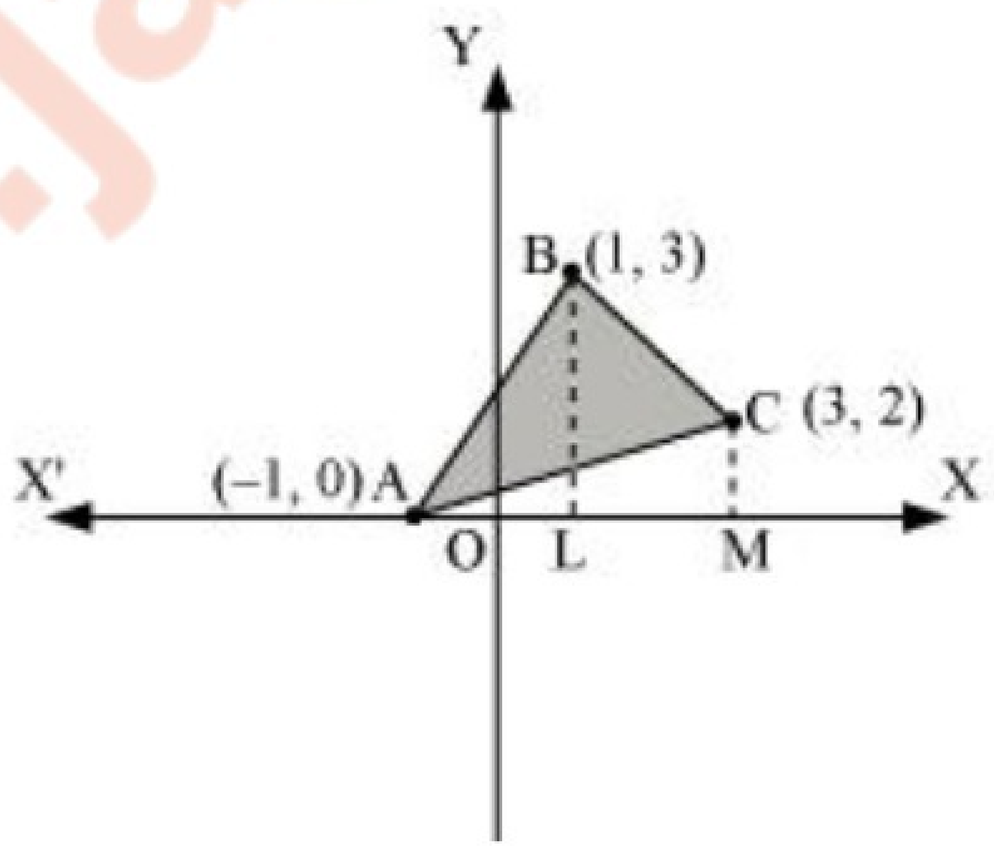
$$= \frac{21}{2} units$$

4. Using integration finds the area of the region bounded by the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

Sol.

From the figure,

Area (ΔACB) = Area (ALBA) + Area (BLMCB) – Area (AMCA)



BL and CM are drawn perpendicular to x-axis.

Equation of line AB is

$$y-0=\frac{3-0}{1+1}(x+1)$$
$$y=\frac{3}{2}(x+1)$$

Area ALBA is

$$=\int_{-1}^{1}\frac{3}{2}(x+1)dx$$
$$=\frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}$$
$$=\frac{3}{2}\left[\frac{1}{2}+1-\frac{1}{2}+1\right]$$
$$=3 \text{ units}$$

Equation of BC is

$$y-3=\frac{2-3}{3-1}(x-1)$$
$$y=\frac{1}{2}(-x+7)$$

Area of region BLMCB is

$$=\int_{1}^{3}\frac{1}{2}(-x+7)dx$$
$$=\frac{1}{2}\left[-\frac{x^2}{2}+7x\right]_{1}^{3}$$
$$=\frac{1}{2}\left[-\frac{9}{2}+21+\frac{1}{2}-7\right]$$
$$=\frac{1}{2}(10)$$
$$=5 \text{ units}$$

Equation of line AC is

$$y - 0 = \frac{2-0}{3+1}(x+1)$$

$$y = \frac{1}{2}(x+1)$$

Area AMCA is

$$= \int_{-1}^{3} \frac{1}{2}(x+1)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{3}$$

$$= \frac{1}{2}\left[\frac{9}{2} + 3 - \frac{1}{2} + 1\right]$$

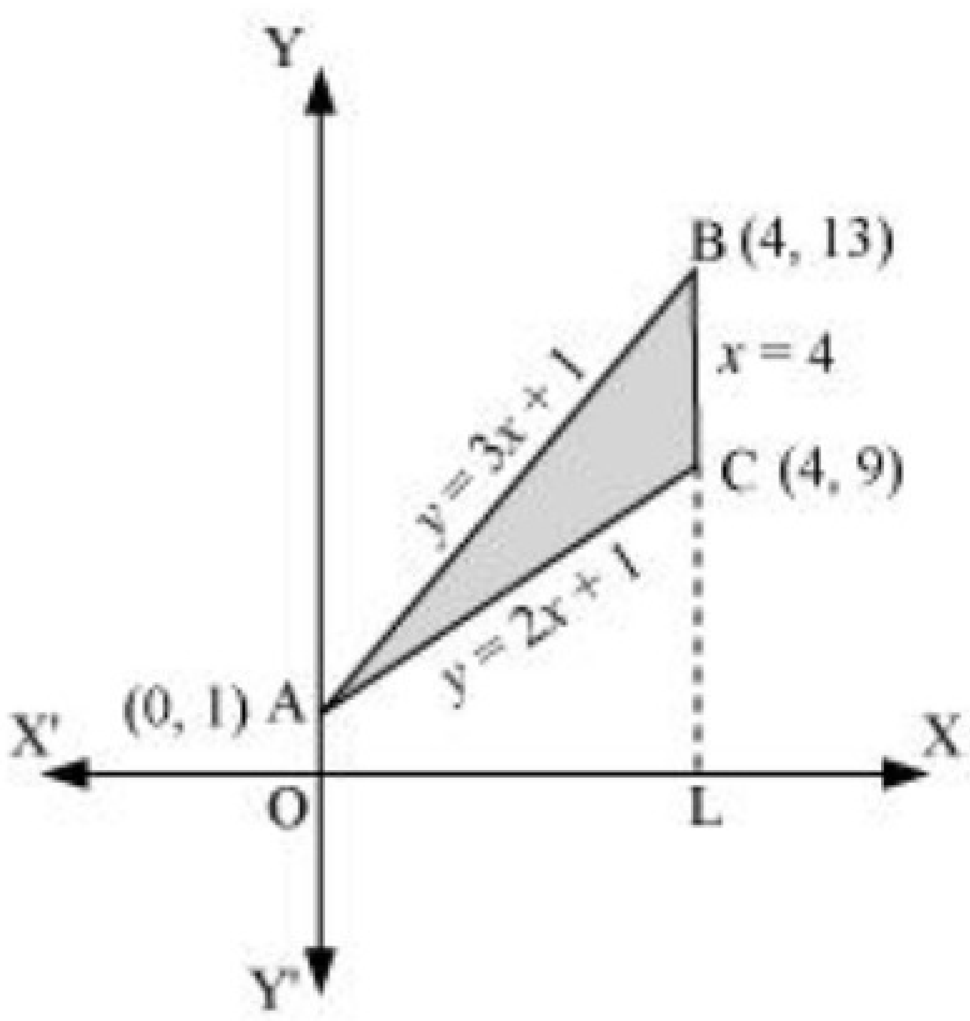
$$= \frac{1}{2}[8]$$

$$= 4 \text{ units}$$

Hence, Area (ΔABC) = (3 + 5 – 4) = 4 units

5. Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Sol.



The equations of sides of the triangle are:

$y = 2x + 1$

$y = 3x + 1$

$x = 4$

On solving these equations, we obtain the vertices of triangle as

A = (0, 1)

B = (4, 13)

C = (4, 9)

We can observe that,

Area (ΔACB) = Area (OLBAO) –Area (OLCAO)

$$= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$$

$$= \left[\frac{3x^2}{2} + x - \frac{2x^2}{2} - x\right]_0^4$$

$$= \left[\frac{x^2}{2}\right]_0^4$$

= 8 units

6. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

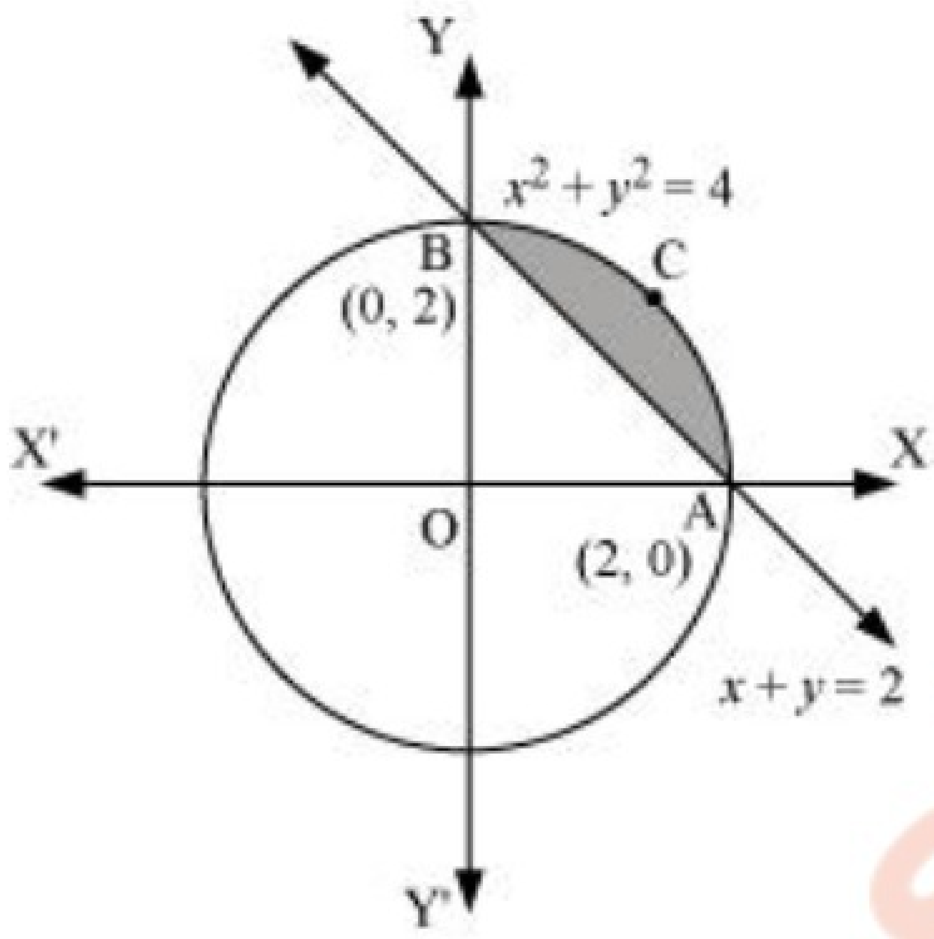
A. 2 (π – 2)

B. π – 2

C. 2π – 1

D. 2 (π + 2)

Sol.



The smaller area enclosed by the circle, is represented by the shaded area ACBA.

Thus,

Area ACBA = Area OACBO – Area (ΔOAB)

$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[2.\frac{\pi}{2}\right] - (4-2)$$

$$= (\pi - 2)\,units$$

Thus, the correct answer is B.

7. Area lying between the curve y2 = 4x and y = 2x is

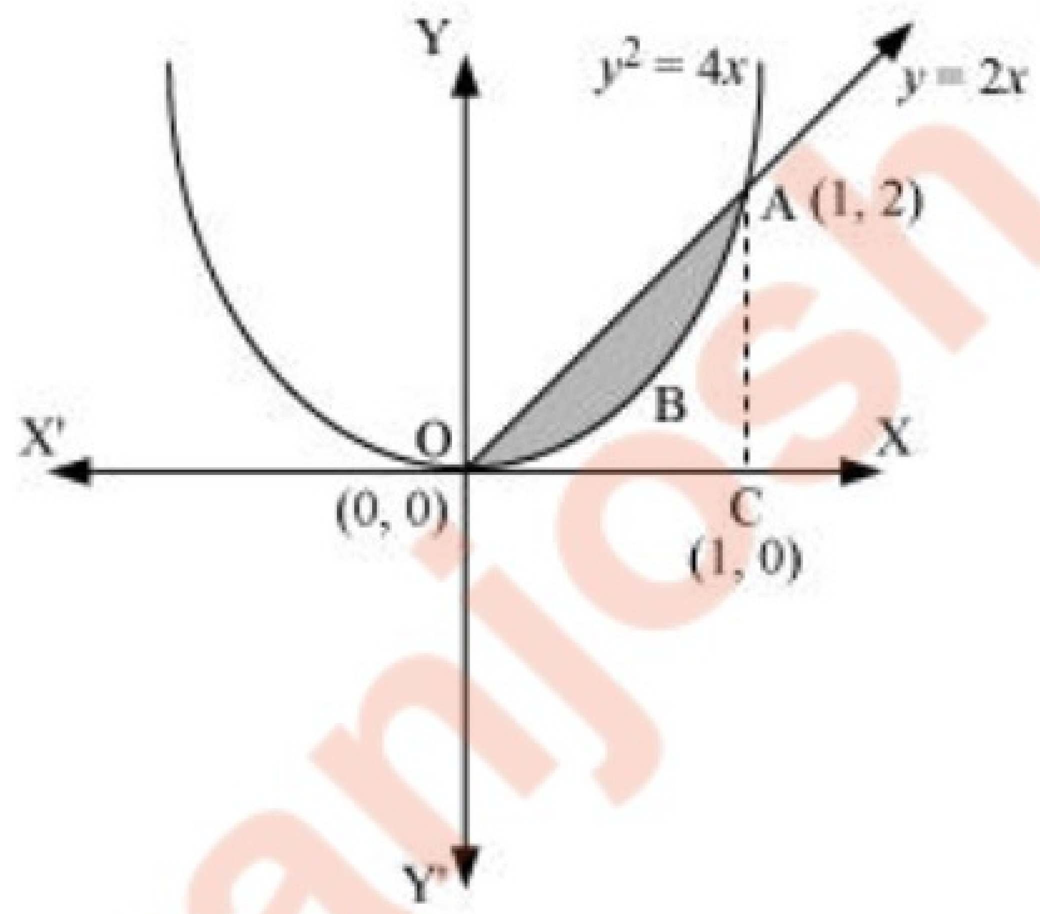
A. $\frac{2}{3}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{3}{4}$

Sol.



The area lying between the curves is represented by the shaded area OBAO.

Construction: - Draw AC perpendicular to x-axis.

Thus,

Area OBAO = Area (ΔOCA) – Area (OCABO)

$$= \int_0^1 2x\,dx - \int_0^1 2\sqrt{x}\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 - 4\left[\frac{x^{\frac{3}{2}}}{3}\right]_0^1$$

$$= \left|1 - \frac{4}{3}\right|$$

$$= \left|-\frac{1}{3}\right|$$

$$= \frac{1}{3} units$$

Hence, the correct answer is B.