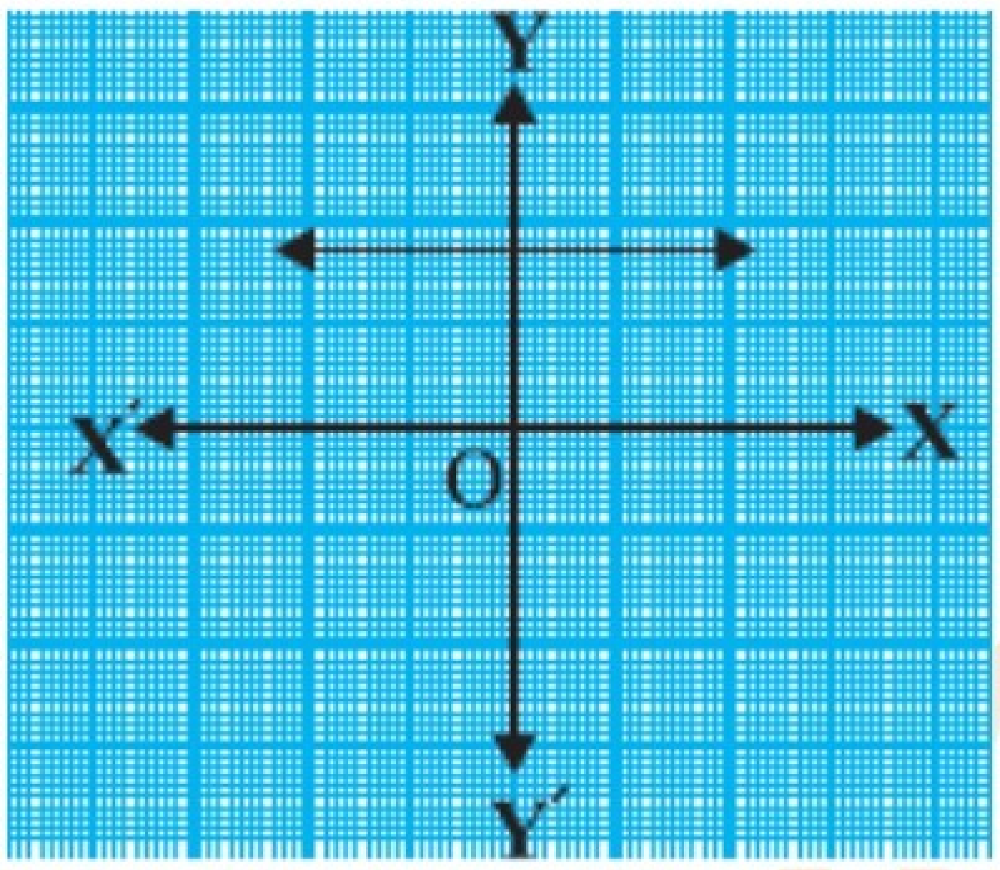
Chapter 2
Polynomials

Exercise: 2.1

Question 1: Find the number of zeroes of $p(x)$, in each case.

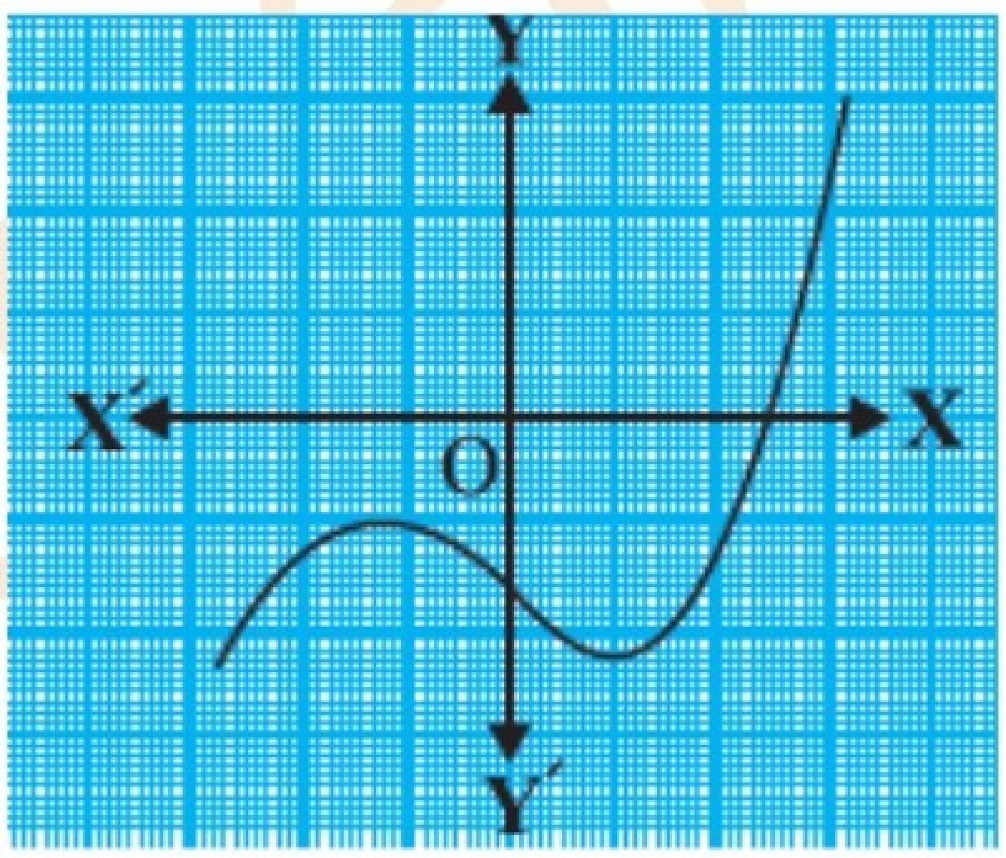
Solution:

(i)



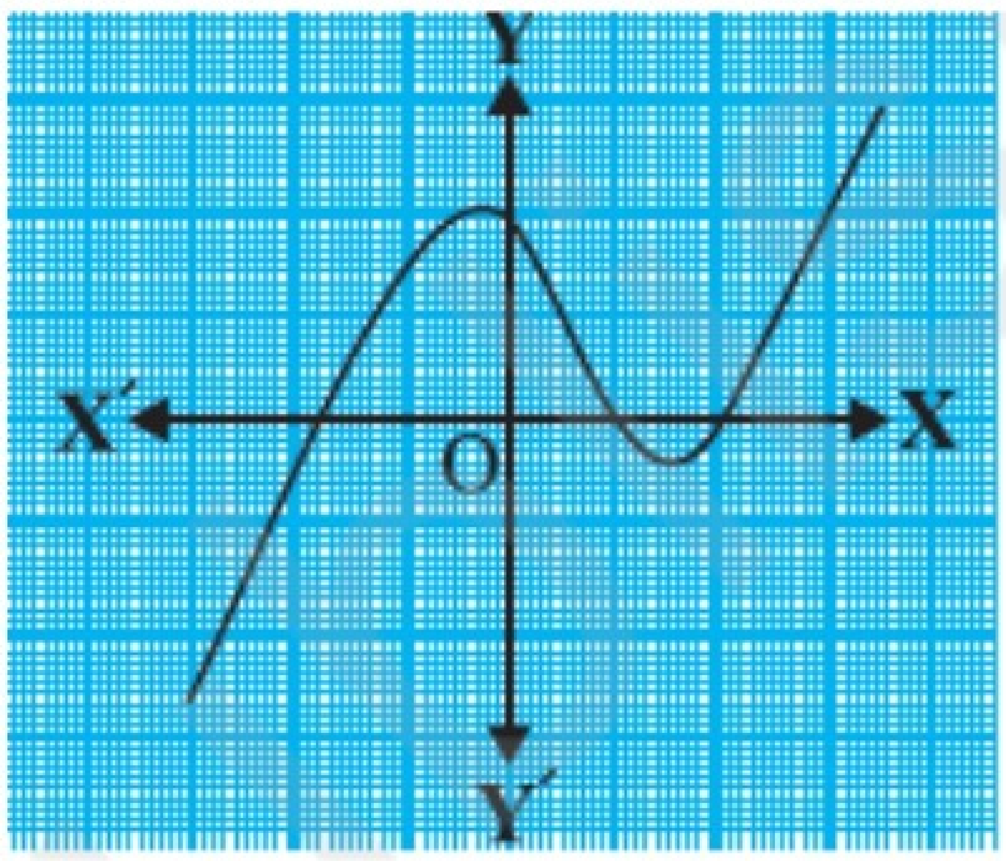
No. of zeroes =0 as graph doesn't intersect at x-axis

(ii)



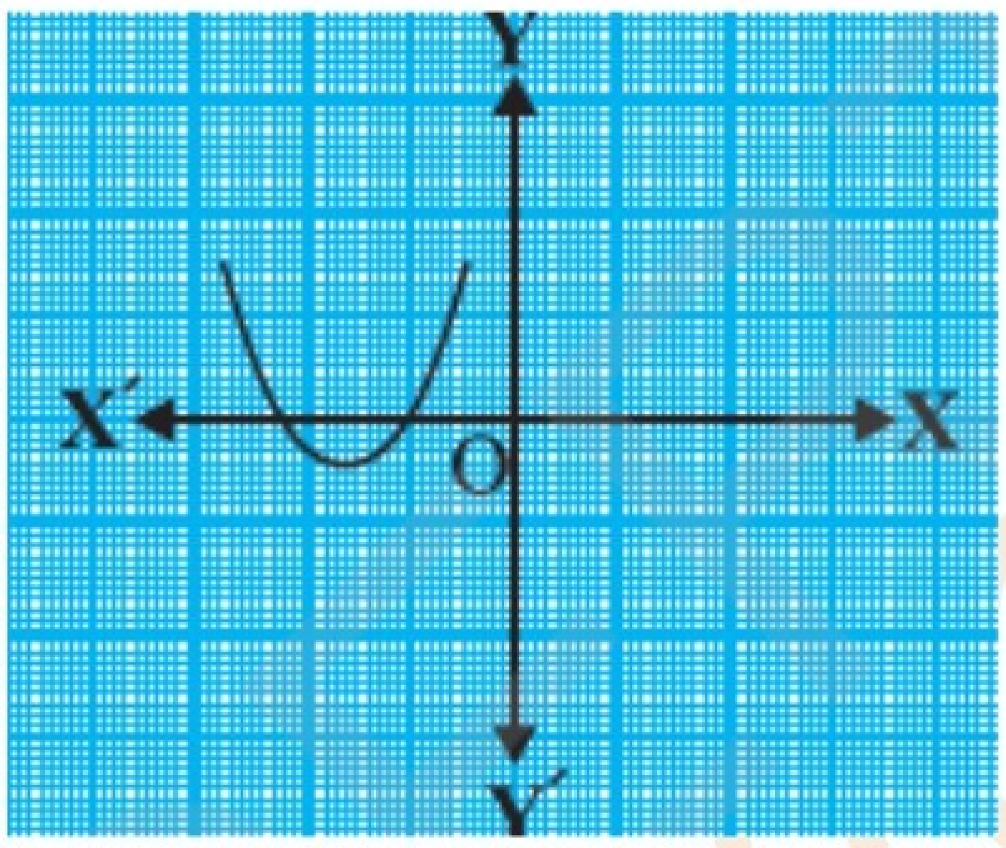
No. of zeroes =1 as graph intersect x-axis once

(iii)



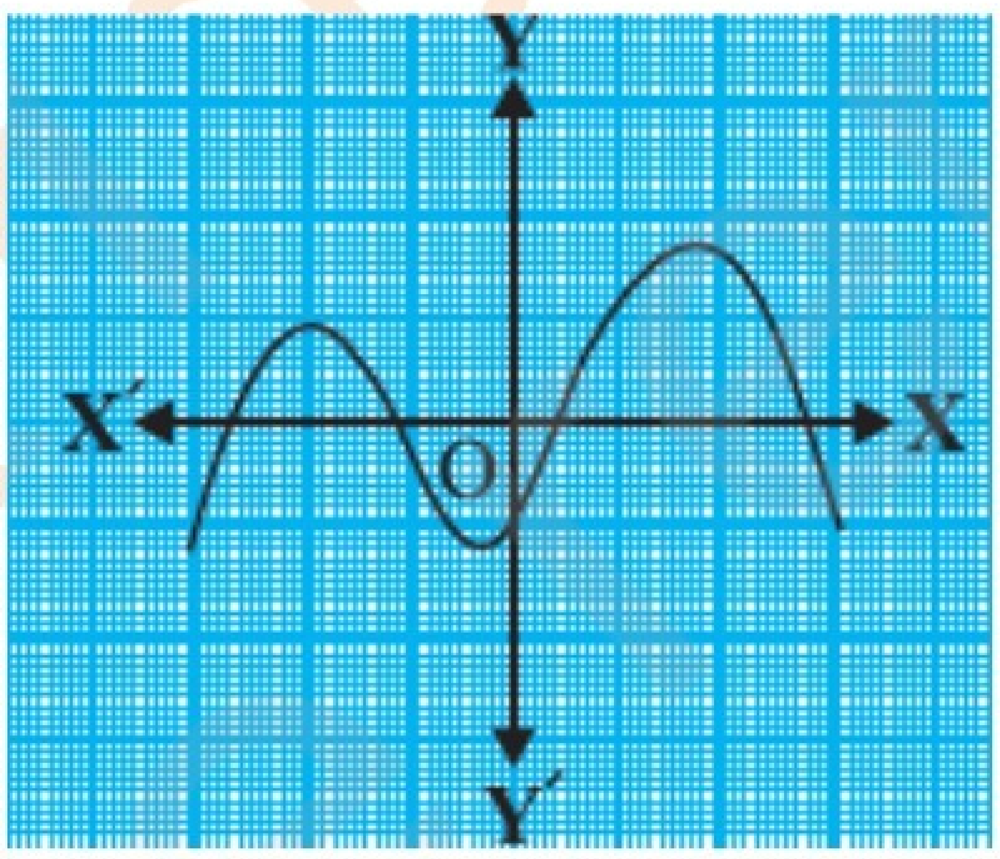
No. of zeroes =3 as graph intersect x-axis three times.

(iv)



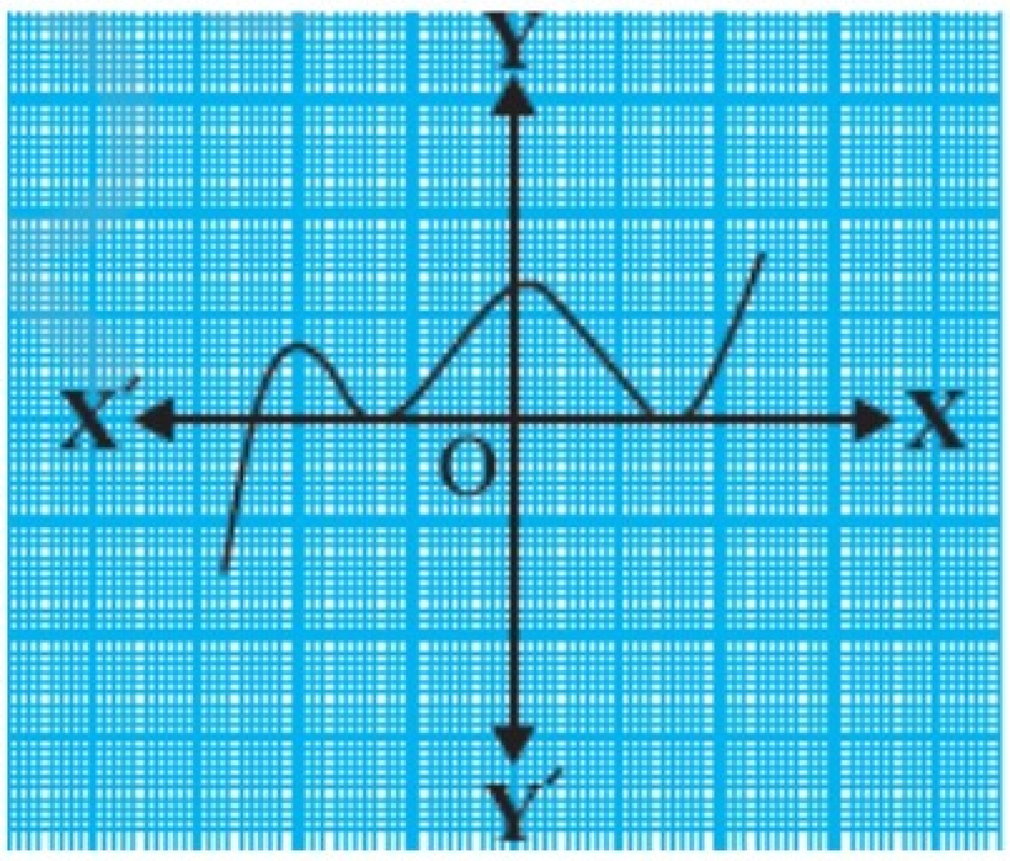
No. of zeroes =2 as graph intersect x-axis two times.

(v)



No. of zeroes =4 as graph intersect x-axis four times.

(vi)



No. of zeroes =0 as graph doesn't intersect at x-axis

Exercise: 2.2

Question 1: Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

Solution:

(i)

$$x^2 - 2x - 8$$

Let

$$x^2 - 2x - 8 = 0$$
$$x^2 - 4x + 2x - 8 = 0$$
$$x(x-4) + 2(x-4) = 0$$
$$(x-4)(x+2) = 0$$
$$x = 4, -2$$

General equation can be represented as:

$$ax^2 + bx + c = 0$$
$$x^2 - 2x - 8 = 0$$
$$a = 1, b = -2, c = -8$$

Now, we will verify the roots

$$\alpha+\beta=-\frac{b}{a}$$
$$4-2=-\frac{(-2)}{1}$$
$$2=2$$

(ii)

$$4s^2-4s+1$$

Let

$$4s^2-4s+1=0$$
$$4s^2-2s-2s+1=0$$
$$2s(2s-1)-1(2s-1)=0$$
$$(2s-1)(2s-1)=0$$
$$s=\frac{1}{2},\frac{1}{2}$$

General equation can be represented as:

$$as^2+bs+c=0$$
$$4s^2-4s+1=0$$
$$a=4,b=-4,c=1$$

Now, we will verify the roots

$$\alpha+\beta=-\frac{b}{a}$$
$$\frac{1}{2}+\frac{1}{2}=-\frac{(-4)}{4}$$
$$1=1$$

(iii)

$$6x^2-3-7x$$

On rearanging the equation:

$$6x^2-7x-3$$

Let

$$6x^2-7x-3=0$$
$$6x^2-9x+2x-3=0$$
$$3x(2x-3)+1(2x-3)=0$$

$$(3x+1)(2x-3)=0$$
$$x=-\frac{1}{3},\frac{3}{2}$$

General equation can be represented as:

$$ax^2+bx+c=0$$
$$6x^2-7x-3=0$$
$$a=6,b=-7,c=-3$$

Now, we will verify the roots

$$\alpha+\beta=-\frac{b}{a}$$
$$-\frac{1}{3}+\frac{3}{2}=-\frac{(-7)}{6}$$
$$\frac{-2+9}{6}=\frac{7}{6}$$
$$\frac{7}{6}=\frac{7}{6}$$

(iv)

$$4u^2+8u$$

Let

$$4u^2+8u=0$$
$$4u(u+2)=0$$
$$u=0,-2$$

General equation can be represented as:

$$au^2+bu+c=0$$
$$4u^2+8u=0$$
$$a=4,b=8,c=0$$

Now, we will verify the roots

$$\alpha+\beta=-\frac{b}{a}$$
$$0-2=-\frac{(8)}{4}$$
$$-2=-2$$

(v)

$$t^2 - 15$$

$$\begin{aligned} &Let \\ &t^2 - 15 = 0 \\ &t^2 = 15 \\ &t = \pm\sqrt{15} \end{aligned}$$

General equation can be represented as:

$$\begin{aligned} &at^2 + bt + c = 0 \\ &t^2 - 15 = 0 \\ &a = 1, b = 0, c = -15 \end{aligned}$$

Now, we will verify the roots

$$\begin{aligned} &\alpha + \beta = -\frac{b}{a} \\ &\sqrt{15} - \sqrt{15} = -\frac{0}{1} \\ &0 = 0 \end{aligned}$$

(vi)

$$3x^2 - x - 4$$

$$\begin{aligned} &Let \\ &3x^2 - x - 4 = 0 \\ &3x^2 - 4x + 3x - 4 = 0 \\ &x(3x - 4) + 1(3x - 4) = 0 \\ &(3x - 4)(x + 1) = 0 \\ &x = -1, \frac{4}{3} \end{aligned}$$

General equation can be represented as:

$$\begin{aligned} &ax^2 + bx + c = 0 \\ &3x^2 - x - 4 = 0 \\ &a = 3, b = -1, c = -4 \end{aligned}$$

Now, we will verify the roots

$$\alpha+\beta=-\frac{b}{a}$$

$$-1+\frac{4}{3}=-\frac{(-1)}{3}$$

$$\frac{-3+4}{3}=\frac{1}{3}$$

$$\frac{1}{3}=\frac{1}{3}$$

Question 2: Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

Solution:

(i)

Let $\alpha+\beta$ are the zeroes of a quadratic polynomial

Now,

We have,

$$\alpha+\beta=\frac{1}{4},\quad \alpha\beta=-1$$

$$\alpha+\beta=\frac{1}{4}$$

$$\frac{-b}{a}=\frac{1}{4}$$

$$\frac{b}{a}=-\frac{1}{4}$$

$$\alpha\beta=-1$$

$$\frac{c}{a}=-1$$

$$\frac{c}{a}=-\frac{4}{4}$$

Thus,

$$a=-4,\ b=1,\ c=4$$

Equation is:

$$-4x^2 + x + 4 = 0$$
$$4x^2 - x - 4 = 0$$

(ii)

Let $\alpha + \beta$ are the zeroes of a quadratic polynomial

Now,

We have,

$$\alpha + \beta = \sqrt{2}, \quad \alpha\beta = \frac{1}{3}$$
$$\alpha + \beta = \sqrt{2}$$
$$-\frac{b}{a} = \sqrt{2}$$
$$\frac{b}{a} = -\frac{3\sqrt{2}}{3}$$

$$\alpha\beta = \frac{1}{3}$$
$$\frac{c}{a} = \frac{1}{3}$$

Thus,

$$a = 3, b = -3\sqrt{2}, c = 1$$

Equation is

$$3x^2 - 3\sqrt{2} + 1 = 0$$

(iii)

Let $\alpha + \beta$ are the zeroes of a quadratic polynomial

Now,

We have,

$$\alpha + \beta = 0, \quad \alpha\beta = \sqrt{5}$$
$$\alpha + \beta = 0$$
$$\frac{-b}{a} = 0$$

$$\frac{-b}{a}=\frac{0}{1}$$

$$\alpha\beta=\sqrt{5}$$

$$\frac{c}{a}=\frac{\sqrt{5}}{1}$$

Thus,

$$a=1, b=0, c=\sqrt{5}$$

Equation is

$$x^2+0x+\sqrt{5}=0$$

$$x^2+\sqrt{5}=0$$

(iv)

Let $\alpha+\beta$ are the zeroes of a quadratic polynomial

Now,

We have,

$$\alpha+\beta=1, \quad \alpha\beta=1$$

$$\alpha+\beta=1$$

$$\frac{-b}{a}=1$$

$$\frac{b}{a}=-\frac{1}{1}$$

$$\alpha\beta=1$$

$$\frac{c}{a}=\frac{1}{1}$$

Thus,

$$a=1, b=-1, c=1$$

Equation is

$$x^2-x+1=0$$

(v)

Let $\alpha+\beta$ are the zeroes of a quadratic polynomial

Now,

We have,

$$\alpha+\beta=-\frac{1}{4}, \quad \alpha\beta=\frac{1}{4}$$

$$\alpha+\beta=-\frac{1}{4}$$

$$\frac{-b}{a}=-\frac{1}{4}$$

$$\frac{b}{a}=\frac{1}{4}$$

$$\alpha\beta=\frac{1}{4}$$

$$\frac{c}{a}=\frac{1}{4}$$

Thus,

$$a=4, b=1, c=1$$

Equation is

$$4x^2+x+1=0$$

(vi)

Let $\alpha+\beta$ are the zeroes of a quadratic polynomial

Now,

We have,

$$\alpha+\beta=4, \quad \alpha\beta=1$$

$$\alpha+\beta=4$$

$$\frac{-b}{a}=4$$

$$\frac{b}{a}=-\frac{4}{1}$$

$$\alpha\beta=1$$

$$\frac{c}{a}=\frac{1}{1}$$

Thus,

$$a = 1, b = -4, c = 1$$

Equation is

$$x^2 - 4x + 1 = 0$$

Exercise: 2.3

Question 1: Divide the polynomial $p(x)$ by the polynomial $g(x)$ and find the quotient and remainder in each of the following:

Solution:

(i)

$$p(x) = x^3 - 3x^2 + 5x - 3$$
$$g(x) = x^2 - 2$$

$$\begin{array}{r|l} & x - 3 \\ x^2 - 2 & x^3 - 3x^2 + 5x - 3 \\ & {}_{-}x^3 \qquad {}_{+}{-2x} \\ \hline & -3x^2 + 7x - 3 \\ & {}_{+}{-3x^2} \qquad {}_{-}{+6} \\ \hline & 7x - 9 \end{array}$$

$$\text{Quotient} = x - 3$$
$$\text{Reminder} = 7x - 9$$

(ii)

$$p(x) = x^4 - 3x^2 + 4x + 5$$
$$g(x) = x^2 + 1 - x$$

$$
\begin{array}{r}
x^2 + x - 3 \\
x^2 + 1 - x \overline{\big) x^4 - 3x^2 + 4x + 5} \\
{}_{-}x^4 \;{}_{-}{+x^2} \qquad {}_{+}{-x^3} \\
\hline
x^3 - 4x^2 + 4x + 5 \\
{}_{-}x^3 \;{}_{+}{-x^2} \;{}_{-}{+x} \\
\hline
-3x^2 + 3x + 5 \\
{}_{+}{-3x^2} \;{}_{-}{+3x} \;{}_{+}{-3} \\
\hline
8
\end{array}
$$

Quotient $= x^2 + x - 3$

Reminder $= 8$

(iii)

$$p(x) = x^4 - 5x + 6$$

$$g(x) = 2 - x^2 = -x^2 + 2$$

$$
\begin{array}{r}
-x^2 - 2 \\
-x^2 + 2 \overline{\big) x^4 - 5x + 6} \\
{}_{-}x^4 \qquad {}_{+}{-2x^2} \\
\hline
2x^2 - 5x + 6 \\
{}_{-}2x^3 \qquad {}_{+}{-4} \\
\hline
-5x + 10
\end{array}
$$

Quotient $= -x^2 - 2$

Reminder $= -5x + 10$

Question 2: Check whether the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial:

Solution:

(i)

$$p(x)=t^2-3$$
$$g(x)=2t^4+3t^3-2t^2-9t-12$$

$$\begin{array}{r|l} & 2t^2+4+3t \\ \hline t^2-3 & 2t^4+3t^3-2t^2-9t-12 \\ & \underline{{}_{-}2t^4 \quad {}_{+}-6t^2} \\ & 4t^2+3t^3-9t-12 \\ & \underline{{}_{-}4t^2 \quad {}_{+}-12} \\ & 3t^3-9t \\ & \underline{{}_{-}3t^3 \; {}_{+}-9t} \\ & 0 \end{array}$$

$$\text{Quotient} = 2t^2+4+3t$$
$$\text{Reminder} = 0$$

Thus, first polynomial is a factor of the second polynomial.

(ii)

$$p(x)=x^2+3x+1$$
$$g(x)=3x^4+5x^3-7x^2+2x+2$$

$$x^2+3x+1\overline{\left)3x^4+5x^3-7x^2+2x+2\right.}\quad 3x^2-4x-4$$

$$\underset{-}{}3x^4\ \underset{-}{}+9x^3+\underset{-}{}9x^2$$

$$-4x^3-16x^2+2x+2$$

$$\underset{+}{}-4x^3\ \underset{+}{}-12x^2\ \underset{+}{}-4x$$

$$-4x^2+6x+2$$

$$\underset{+}{}-4x^2\ \underset{+}{}-12x\ \underset{+}{}-4$$

$$18x+6$$

Quotient $=3x^2-4x-4$

Reminder $=18x+6$

Thus, first polynomial is not a factor of the second polynomial

(iii)

$$p(x)=x^2-3x+1$$

$$g(x)=x^5-4x^3+x^2+3x+1$$

$$x^2-3x+1\overline{\left)x^5-4x^3+x^2+3x+1\right.}\quad x^3+3x^2-4x-14$$

$$\underset{-}{}x^5\ \underset{-}{}+x^3\qquad \underset{+}{}-3x^4$$

$$3x^4-5x^3+x^2+3x+1$$

$$\underset{-}{}3x^4\ \underset{+}{}-9x^3\ \underset{-}{}+3x^2$$

$$-4x^3-2x^2+3x+1$$

$$\underset{+}{}-4x^3\ \underset{-}{}+12x^2\ \underset{+}{}-4x$$

$$-14x^2+7x+1$$

$$\underset{+}{}-14x^2\ \underset{-}{}+42x\ \underset{+}{}-14$$

$$-35x+15$$

$$\text{Quotient} = x^3 + 3x^2 - 4x - 14$$
$$\text{Reminder} = -35x + 15$$

Thus, first polynomial is not a factor of the second polynomial.

Question 3: Obtain all other zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

Solution:

$$p(x) = 3x^4 + 6x^3 - 2x^2 - 10x - 5$$

Now, $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$ are the two zeroes of the given polynomial.

Thus, $\left(x - \sqrt{\frac{5}{3}}\right)\left(x + \sqrt{\frac{5}{3}}\right) = \left(x^2 - \frac{5}{3}\right)$ is a factor of given polynomial

Let $\quad g(x) = \left(x^2 - \frac{5}{3}\right)$

On division,

$$\begin{array}{r|l}
 & 3x^2 + 6x + 3 \\
\hline
x^2 - \frac{5}{3} & 3x^4 + 6x^3 - 2x^2 - 10x - 5 \\
 & {}_{-}3x^4 \quad {}_{+}-5x^2 \\
\hline
 & 6x^3 + 3x^2 - 10x - 5 \\
 & {}_{-}6x^3 \quad {}_{+}-10x \\
\hline
 & 3x^2 - 5 \\
 & {}_{-}3x^2 \; {}_{+}-5 \\
\hline
 & 0
\end{array}$$

Thus,

$$3x^4+6x^3-2x^2-10x-5=\left(x^2-\frac{5}{3}\right)\left(3x^2+6x+3\right)$$

$$3x^4+6x^3-2x^2-10x-5=3\left(x^2-\frac{5}{3}\right)\left(x^2+2x+1\right)$$

$$3x^4+6x^3-2x^2-10x-5=3\left(x-\sqrt{\frac{5}{3}}\right)\left(x+\sqrt{\frac{5}{3}}\right)\left(x+1\right)^2$$

Hence,

Zeroes of the polynomial are $=\sqrt{\frac{5}{3}},-\sqrt{\frac{5}{3}},-1,-1$

Question 4: On dividing x^3-3x^2+x+2 by a polynomial $g(x)$, the quotient and remainder were $x-2$ and $-2x+4$, respectively. Find $g(x)$.

Solution:

$p(x)=x^3-3x^2+x+2$

$g(x)=?$

Quotient $=x-2$

Remainder $=-2x+4$

We know that,

$p(x)=g(x)\times$ Quotient + Remainder

$x^3-3x^2+x+2 \;=g(x)\times(x-2)-2x+4$

$x^3-3x^2+x+2+2x-4=g(x)\times(x-2)$

$g(x)\times(x-2)=x^3-3x^2+3x-2$

$$g(x)=\frac{x^3-3x^2+3x-2}{x-2}$$

Now,

$$\begin{array}{r l} & x^2-x+1 \\ x-2 \big) & x^3-3x^2+3x-2 \\ & {}_{-}x^3 {}_{+}{-2x^2} \end{array}$$

$$\begin{array}{l} -x^2+3x-2 \\ {}_{+}-x^2\,{}_{-}+2x \\ \hline x-2 \\ {}_{-}x\,{}_{+}-2 \\ \hline 0 \end{array}$$

Thus,

$g(x)=x^2-x+1$

Exercise: 2.4

Question 1: Verify that the numbers given alongside of the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case:

Solution:

(i)

$$2x^3+x^2-5x+2 \;\; ; \;\; \frac{1}{2},1,-2$$

On comparing coefficients, we got

$$a=2, b=1, c=-5, d=2$$

Let,

$$y=2x^3+x^2-5x+2$$

$$y\left(x=\frac{1}{2}\right)=2.\frac{1}{8}+\frac{1}{4}-\frac{5}{2}+2=0$$

$$y(x=1)=2+1-5+2=0$$

$$y(x=-2)=-16+4+10+2=0$$

Thus, all three values of x are the zeroes of the given polynomial

Now,

$$\alpha+\beta+\gamma=-\frac{b}{a}$$
$$\frac{1}{2}+1-2=-\frac{1}{2}$$
$$-\frac{1}{2}=-\frac{1}{2}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$$
$$\frac{1}{2}-2-1=-\frac{5}{2}$$
$$-\frac{5}{2}=-\frac{5}{2}$$

$$\alpha\beta\gamma=-\frac{d}{a}$$
$$\frac{1}{2}.1.(-2)=-1$$
$$-1=-1$$

(ii)

$$x^3-4x^2+5x-2 \quad ; \quad 2,1,1$$

On comparing coefficients, we got

$$a=1, b=-4, c=5, d=-2$$

Let

$$y=x^3-4x^2+5x-2$$
$$y(x=2)=8-16+10-2=0$$
$$y(x=1)=1-4+5-2=0$$
$$y(x=1)=1-4+5-2=0$$

Thus, all three values of x are the zeroes of the given polynomial

Now,

$$\alpha+\beta+\gamma=-\frac{b}{a}$$
$$2+1+1=4$$
$$4=4$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$$
$$2+1+2=5$$
$$5=5$$

$$\alpha\beta\gamma=-\frac{d}{a}$$
$$2=2$$

Question 2: Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, –7, –14 respectively.

Solution:

Given

$$\alpha+\beta+\gamma=2$$
$$\alpha\beta+\beta\gamma+\alpha\gamma=-7$$
$$\alpha\beta\gamma=-14$$

$$\alpha+\beta+\gamma=2$$
$$-\frac{b}{a}=\frac{2}{1}$$

$$\alpha\beta+\beta\gamma+\alpha\gamma=-7$$
$$\frac{c}{a}=\frac{-7}{1}$$

$$\alpha\beta\gamma = -14$$
$$\frac{d}{a} = \frac{-14}{1}$$

We got,

$$a = 1, b = -2, c = -7, d = -14$$

Thus, polynomial will be:

$$ax^3 + bx^2 + cx + d = 0$$
$$x^3 - 2x^2 - 7x - 14 = 0$$

Question 3: If the zeroes of the polynomial $x^3 - 3x^2 + x + 1$ are $a - b$, a, $a + b$, find a and b.

Solution:

Let

$$y = x^3 - 3x^2 + x + 1$$

On comparing,

$$a = 1, b = -3, c = 1, d = 1$$

Zeroes $= a - b, a, a + b$

$$\alpha + \beta + \gamma = -\frac{b}{a}$$
$$a - b + a + a + b = 3$$
$$3a = 3$$
$$a = 1$$
$$\alpha\beta\gamma = -\frac{d}{a}$$
$$a(a-b)(a+b) = -1$$
$$a(a^2 - b^2) = -1$$
$$1(1 - b^2) = -1$$
$$1 - b^2 = -1$$
$$b^2 = 2 \Rightarrow b = \pm\sqrt{2}$$

Thus,

$$a = 1, b = \pm\sqrt{2}$$

Question 4: If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.

Solution:

Let

$$y = x^4 - 6x^3 - 26x^2 + 138x - 35$$

Zeroes $= 2 \pm \sqrt{3}$

$$\alpha + \beta + \gamma + \delta = 6$$
$$2 + \sqrt{3} + 2 - \sqrt{3} + \gamma + \delta = 6$$
$$\gamma + \delta = 2$$

$$\alpha\beta\gamma\delta = -35$$
$$\left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right)\gamma\delta = -35$$
$$\left(4 - 3\right)\gamma\delta = -35$$
$$\gamma\delta = -35$$

Now,

$$\gamma + \delta = 2$$
$$\gamma = 2 - \delta \text{........................(1)}$$

$$\gamma\delta = -35$$
$$\left(2 - \delta\right)\delta = -35$$
$$2\delta - \delta^2 + 35 = 0$$
$$\delta^2 - 2\delta - 35 = 0$$
$$\delta^2 - 7\delta + 5\delta - 35 = 0$$
$$\delta\left(\delta - 7\right) + 5\left(\delta - 7\right) = 0$$
$$\left(\delta - 7\right)\left(\delta + 5\right) = 0$$
$$\delta = 7, -5$$

$$\gamma = 2 - \delta$$
$$\gamma = 2 - 7 = -5$$

Hence, two other zeroes are 7 and -5.

Question 5: If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k and a.

Solution:

Let

$$y = x^4 - 6x^3 + 16x^2 - 25x + 10$$

Now,

$$x^4 - 6x^3 + 16x^2 - 25x + 10 = \left(x^2 - 2x + k\right)Q + x + a$$

$$\begin{array}{r|l}
 & x^2 - 4x + 8 - k \\
x^2 - 2x + k & x^4 - 6x^3 + 16x^2 - 25x + 10 \\
 & \underline{{}_{-}x^4 \; {}_{+}{-2x^3} \; {}_{-}{+kx^2}} \\
 & -4x^3 + \left(16 - k\right)x^2 - 25x + 10 \\
 & \underline{{}_{+}{-4x^3} \; {}_{-}{+8x^2} \; {}_{+}{-4kx}} \\
 & \left(8 - k\right)x^2 + \left(-25 + 4k\right)x + 10 \\
 & \underline{{}_{-}\left(8 - k\right)x^2 \; {}_{+}{-\left(16 - 2k\right)x} \; {}_{-}{+k\left(8 - k\right)}} \\
 & \left(-9 + 2k\right)x + 10 - 8k + k^2
\end{array}$$

Now,

$$Q = x^2 - 4x + 8 - k$$
$$R = \left(-9 + 2k\right)x + 10 - 8k + k^2$$
$$x + a = \left(-9 + 2k\right)x + 10 - 8k + k^2$$

$$1 = -9 + 2k$$
$$2k = 10$$
$$k = 5$$

$$a = 10 - 8k + k^2$$
$$a = 10 - 40 + 25$$
$$a = -5$$

Thus,

$$a = -5, k = 5$$