Mathematics Class 12th NCERT Solutions

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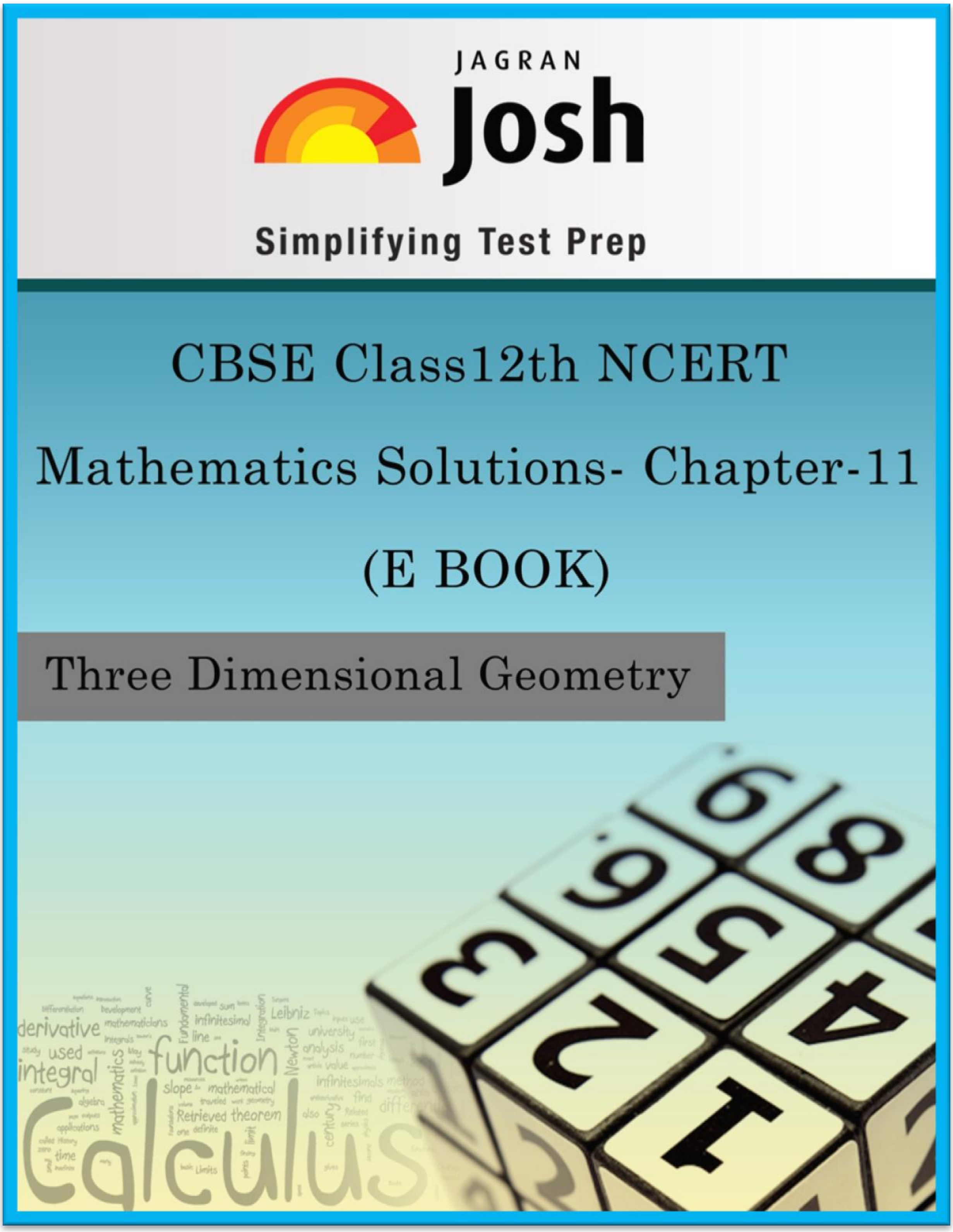
Simplifying Test Prep



Preface

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The purpose of providing solutions for CBSE class 12th Science and Mathematics NCERT book is to explain the questions in an easy way and as per the CBSE marking scheme. This is a product exclusively for CBSE class 12th students which acts as a time-saver by providing a pattern for the solutions of NCERT based questions as per the CBSE curriculum. This document help to build a strong concept on the chapter mentioned in here and hence the students have been guided in the most appropriate way for their board examination

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Chapter -11
Three Dimensional Geometry
Class – XII
Subject –Maths

Exersise-11.1

1. If a line makes angles 90^0, 135^0, 45^0 with x, y and z-axes respectively, find its direction cosines.

Sol.

Let direction cosines of line are $= l, m \ \& \ n$

$$l = \cos 90^\circ = 0$$

$$m = \cos 135^\circ = -\frac{1}{\sqrt{2}}$$

$$n = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

2. Find the direction cosines of a line which makes equal angles with the coordinate axes.

Sol.

Let direction cosines of line make angle α

$l = \cos\alpha$

$m = \cos\alpha$

$n = \cos\alpha$

$l^2 + m^2 + n^2 = 1$

$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$3\cos^2\alpha = 1$

$\cos^2\alpha = \frac{1}{3}$

$\cos\alpha = \pm\frac{1}{\sqrt{3}}$

3. If a line has the direction ratios -18, 12, -4, then what are its direction cosines?

Sol.

Direction ration are $= -18, 12 \ \& \ -4$

Direction cosines are:

$$= \frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$

$$= \frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}$$

$$= \frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$$

4. Show that the points (2, 3, 4), (-1, -2, 1), (5 8, 7) are collinear.

Sol.

Points are:

$A = (2, 3, 4)$

$B = (-1, -2, 1)$

$C = (5, 8, 7)$

Direction ratios of AB $= (-1-2), (-2-3), (1-4)$

$= -3, -5, -3$

Direction ratios of BC $= (5+1), (8+2), (7-1)$

$= 6, 10, 6$

Direction ratios of BC $= -2 \times$ Direction ratios of AB

Thus, AB is parallel to BC. A, B & C are collinear points.

5. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, -4), (- 1, 1, 2) and (- 5,-5, -2)

Sol.

Vertices of triangle are:

$A = (3, 5, -4)$

$B = (-1, 1, 2)$

$C = (-5, -5, -2)$

Direction ratios of AB are:

$= (-1-3),(1-5),(2+4)$

$= -4,-4,6$

Direction cosines of AB are:

$$= \frac{-4}{\sqrt{(-4)^2+(-4)^2+6^2}}, \frac{-4}{\sqrt{(-4)^2+(-4)^2+6^2}}, \frac{6}{\sqrt{(-4)^2+(-4)^2+6^2}}$$

$$= \frac{-4}{\sqrt{68}}, \frac{-4}{\sqrt{68}}, \frac{6}{\sqrt{68}}$$

$$= \frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$$

Direction ratios of BC are:

$= (-5+1),(-5-1),(-2-2)$

$= -4,-6,-4$

Direction cosines of BC are:

$$=\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}},\frac{-6}{\sqrt{(-4)^2+(-6)^2+(-4)^2}},\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$$

$$=\frac{-4}{\sqrt{68}},\frac{-6}{\sqrt{68}},\frac{-4}{\sqrt{68}}$$

$$=\frac{-2}{\sqrt{17}},\frac{-3}{\sqrt{17}},\frac{-2}{\sqrt{17}}$$

Direction ratios of CA are:

$$=(-5-3),(-5-5),(-2+4)$$

$$=-8,-10,2$$

Direction cosines of CA are:

$$=\frac{-8}{\sqrt{(-8)^2+(-10)^2+(2)^2}},\frac{-10}{\sqrt{(-8)^2+(-10)^2+(2)^2}},\frac{2}{\sqrt{(-8)^2+(-10)^2+(2)^2}}$$

$$=\frac{-8}{2\sqrt{42}},\frac{-10}{2\sqrt{42}},\frac{2}{2\sqrt{42}}$$

$$=\frac{-4}{\sqrt{42}},\frac{-5}{\sqrt{42}},\frac{1}{\sqrt{42}}$$

Exercise-11.2

1. Show that the three lines with direction cosines

$$\frac{12}{13},\frac{-3}{13},\frac{-4}{13};\frac{4}{13},\frac{12}{13},\frac{3}{13};\frac{3}{13},\frac{-4}{13},\frac{12}{13}$$ **are mutually perpendicular.**

Sol.

For lines: - $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ & $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$

$$\Rightarrow l_1 l_2 + m_1 m_2 + n_1 n_2$$
$$= \frac{12}{13} \times \frac{4}{13} + \frac{-3}{13} \times \frac{12}{13} + \frac{-4}{13} \times \frac{3}{13}$$
$$= \frac{48}{169} - \frac{36}{169} - \frac{12}{169}$$
$$= 0$$

Hence, this lines are perpendicular

For lines: - $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ & $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$

$$\Rightarrow l_1 l_2 + m_1 m_2 + n_1 n_2$$
$$= \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \frac{-4}{13} + \frac{3}{13} \times \frac{12}{13}$$
$$= \frac{12}{169} - \frac{48}{169} + \frac{36}{169}$$
$$= 0$$

Hence, this lines are perpendicular

For lines: - $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ & $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$

$$\Rightarrow l_1 l_2 + m_1 m_2 + n_1 n_2$$
$$= \frac{12}{13} \times \frac{3}{13} + \frac{-3}{13} \times \frac{-4}{13} + \frac{-4}{13} \times \frac{12}{13}$$
$$= \frac{36}{169} + \frac{12}{169} - \frac{48}{169}$$
$$= 0$$

Hence, this lines are perpendicular

$\therefore$ All lines are mutually perpendicular

2. **Show that the line through the points (1, -1, 2) (3, 4, -2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).**

Sol.

Let $A = (1, -1, 2)$

$B = (3, 4, -2)$

$C = (0, 3, 2)$

$D = (3, 5, 6)$

Direction ratios of $AB = (3-1), (4+1), (-2-2)$

$= 2, 5, -4$

Direction ratios of $CD = (3-0), (5-3), (6-2)$

$= 3, 2, 4$

$\Rightarrow a_1a_2 + b_1b_2 + c_1c_2$

$= 2 \times 3 + 5 \times 2 + (-4) \times 4$

$= 6 + 10 - 16$

$= 0$

Thus, AB & CD lines are perpendicular to each other.

3. **Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (-1, -2, 1), (1, 2, 5).**

Sol.

Let $A = (4,7,8)$

$B = (2,3,4)$

$C = (-1,-2,1)$

$D = (1,2,5)$

Direction ratios of $AB = (2-4),(3-7),(4-8)$

$= -2,-4,-4$

Direction ratios of $CD = (1+1),(2+2),(5-1)$

$= 2,4,4$

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \frac{-2}{2} = \frac{-4}{4} = \frac{-4}{4}$$

$$\Rightarrow -1 = -1 = -1$$

Thus, AB is parallel to CD

4. Find the equation of the line which passes through the point (1, 2 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Sol.

Point $A = (1, 2, 3)$

Position vector through point $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

Given vector, $\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$

When a line is passes through a point & parallel to a vector then,

$\vec{r} = \vec{a} + \lambda \vec{b}$

$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} + 2\hat{j} - 2\hat{k}\right)$

This is the required equation of line.

5. Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

Sol.

$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

We know that, equation of line will be:

$\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$

Now,

The equation of line in vector form is:

$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$

$\vec{r} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$

cartesian form of the equation is:

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the equation of line in cartesian form.

6. Find the Cartesian equation of the line which Passes through the Point (-2, 4 -5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Sol.

Point A $= (-2, 4, -5)$

Direction ratios of the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$ is:

$= 3, 5, 6$

Direction ratios of the required line is $= 3k, 5k, 6k$

We know that, the equation of a line passing through the point (x_1, y_1, z_1) & with the direction ratios a,b,c is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Thus, the equtaion of required line will be:

$$\frac{x+2}{3k} = \frac{y-4}{5k} = \frac{z+5}{6k}$$

$$\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6} = k$$

7. The Cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.

Sol.

Cartesian equation of line is:

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

This line is passes through the point $= (5, -4, 6)$

The position vector of this point, $\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$

The direction ratios of this line $= 3, 7, 2$

Thus,

$\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

The line through $\vec{a}$ & in the direction of $\vec{b}$ is:

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda\left(3\hat{i} + 7\hat{j} + 2\hat{k}\right)$

This is the equation of line in vector form.

8. Find the vector and the Cartesian equations of the lines that pass through the origin and (5, -2, 3).

Sol.

The line is passes through origin.

$\therefore \vec{a} = 0$

The direction ratios of the line through origin & $(5,-2,3)$ are $= (5-0),(-2-0),(3-0)$

$= 5,-2,3$

$\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$

Thus, equation of line in vector form is:

$\vec{r} = \vec{a} + \lambda \vec{b}$

$\vec{r} = 0 + \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$

$\vec{r} = \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$

Hence, the equation of required line in cartesian form is

$\frac{x-0}{5} = \frac{y-0}{-2} = \frac{z-0}{3}$

$\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$

9. Find the vector and the Cartesian equations of the line that passes through the points (3, -2, -5), (3, -2, 6)

Sol.

Let point $A = (3,-2,-5)$

$B = (3,-2,6)$

Line AB is passes through A.

Thus,

$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$

The direction ratios of AB $= (3-3), (-2+2), (6+5)$

$= 0, 0, 11$

The equation of vector in the direction of AB is

$\vec{b} = 0\hat{i} + 0\hat{j} + 11\hat{k}$

The equation of AB in vector form is:

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda\left(11\hat{k}\right)$

The equation of AB in cartesian form is:

$$\frac{x-3}{0} = \frac{y+2}{0} = \frac{z+5}{11}$$

10. Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda\left(3\hat{i} - 2\hat{j} + 6\hat{k}\right)$ & $\vec{r} = 7\hat{i} - 6\hat{k} + \mu\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda\left(\hat{i} - \hat{j} - 2\hat{k}\right)$ & $\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$

Sol.

(i)

$\vec{a_1} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\vec{a_2} = \hat{i} + 2\hat{j} + 2\hat{k}$

Angle between the given pair of lines, $\cos\theta = \left|\dfrac{\vec{a_1}.\vec{a_2}}{\left|\vec{a_1}\right|\left|\vec{a_2}\right|}\right|$

$$\vec{a_1}.\vec{a_2} = \left(3\hat{i} - 2\hat{j} + 6\hat{k}\right)\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

$$= 3 \times 1 - 2 \times 2 + 6 \times 2$$

$$= 3 - 4 + 12$$

$$= 11$$

$$\left|\vec{a_1}\right| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$\left|\vec{a_2}\right| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

$$\cos\theta = \frac{19}{7 \times 3} = \frac{11}{21}$$

$$\theta = \cos^{-1}\left(\frac{11}{21}\right)$$

(ii)

$$\vec{a_1} = \hat{i} - \hat{j} - 2\hat{k}$$

$$\vec{a_2} = 3\hat{i} - 5\hat{j} - 4\hat{k}$$

Angle between the given pair of lines, $\cos\theta = \left|\dfrac{\vec{a_1}.\vec{a_2}}{\left|\vec{a_1}\right|\left|\vec{a_2}\right|}\right|$

$$\vec{a_1}.\vec{a_2} = \left(\hat{i} - \hat{j} - 2\hat{k}\right)\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$$

$$= 1 \times 3 - 1 \times (-5) - 2 \times (-4)$$

$$= 3 + 5 + 8$$

$$= 16$$

$$\left|\vec{a_1}\right| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\left|\vec{a_2}\right| = \sqrt{3^2 + 5^2 + 4^2} = 5\sqrt{2}$$

$$\cos\theta = \frac{16}{\sqrt{6}\times 5\sqrt{2}}$$
$$= \frac{16}{10\sqrt{3}}$$
$$= \frac{8}{5\sqrt{3}}$$
$$\theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

11. Find the angle between the following pair of lines:

1) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

2) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Sol.

1)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$$

$$\vec{a_1} = 2\hat{i} + 5\hat{j} - 3\hat{k}$$
$$\vec{a_2} = -\hat{i} + 8\hat{j} + 4\hat{k}$$

Angle between the given pair of lines, $\cos\theta = \left|\frac{\vec{a_1}.\vec{a_2}}{|\vec{a_1}||\vec{a_2}|}\right|$

$$\vec{a_1}.\vec{a_2} = (2\hat{i} + 5\hat{j} - 3\hat{k})(-\hat{i} + 8\hat{j} + 4\hat{k})$$
$$= 2\times -1 + 5\times 8 - 3\times(4)$$
$$= -2 + 40 - 12$$
$$= 26$$

$$\left|\overrightarrow{a_1}\right| = \sqrt{2^2 + 5^2 + (-3)^2} = \sqrt{38}$$

$$\left|\overrightarrow{a_2}\right| = \sqrt{(-1)^2 + 8^2 + 4^2} = 9$$

$$\cos\theta = \frac{26}{\sqrt{38} \times 9}$$

$$= \frac{26}{9\sqrt{38}}$$

$$\theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

2)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1} \text{ and } \frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$$

$$\overrightarrow{a_1} = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\overrightarrow{a_2} = 4\hat{i} + \hat{j} + 8\hat{k}$$

Angle between the given pair of lines, $\cos\theta = \left|\frac{\overrightarrow{a_1}.\overrightarrow{a_2}}{\left|\overrightarrow{a_1}\right|\left|\overrightarrow{a_2}\right|}\right|$

$$\overrightarrow{a_1}.\overrightarrow{a_2} = \left(2\hat{i} + 2\hat{j} + \hat{k}\right)\left(4\hat{i} + \hat{j} + 8\hat{k}\right)$$

$$= 2 \times 4 + 2 \times 1 + 1 \times 8$$

$$= 8 + 2 + 8$$

$$= 18$$

$$\left|\overrightarrow{a_1}\right| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\left|\overrightarrow{a_2}\right| = \sqrt{4^2 + 1^2 + 8^2} = 9$$

$$\cos\theta = \frac{18}{27} = \frac{2}{3}$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

12. Find the values of p so that the line

$$\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2} \text{ and } \frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$$ **are at right angles.**

Sol.

$$\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2} \text{ and } \frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$$

$$\frac{x-1}{-3}=\frac{y-2}{\frac{2p}{7}}=\frac{z-3}{2} \text{ and } \frac{x-1}{-\frac{3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$$

Direction ratios of the above lines will be:

$$a_1=-3,\ b_1=\frac{2p}{7},\ c_1=2$$

$$a_2=\frac{-3p}{7},\ b_2=1,\ c_2=-5$$

Two lines are perpendicular to each other if

$$a_1a_2+b_1b_2+c_1c_2=0$$

$$-3\times\left(\frac{-3p}{7}\right)+\frac{2p}{7}\times 1+2\times(-5)=0$$

$$\frac{9p}{7}+\frac{2p}{7}-10=0$$

$$\frac{11p}{7}=10$$

$$p=\frac{70}{11}$$

13. Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ **and** $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ **are perpendicular to each other.**

Sol.

$$\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1} \text{ and } \frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

Direction ratios of the above lines will be:

$a_1 = 7,\ b_1 = -5,\ c_1 = 1$

$a_2 = 1,\ b_2 = 2,\ c_2 = 3$

Two lines are perpendicular to each other if

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow\ 7\times 1 + (-5)\times 2 + 1\times 3$

$= 7 - 10 + 3$

$= 0$

Thus, the lines are perpendicular to each other.

14. Find the shortest distance between the lines

$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ **and**

$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

Sol.

$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

$\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$

$\vec{b_1} = \hat{i} - \hat{j} + \hat{k}$

$\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$

$\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$

Shortest distance between the above lines, $d = \left| \frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$

$\vec{a_2} - \vec{a_1} = 2\hat{i} - \hat{j} - \hat{k} - (\hat{i} + 2\hat{j} + \hat{k})$

$= \hat{i} - 3\hat{j} - 2\hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$= -3\hat{i} + 3\hat{k}$

$|\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$

Now,

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$d = \left| \frac{(-3\hat{i} + 3\hat{k}).(\hat{i} - 3\hat{j} - 2\hat{k})}{3\sqrt{2}} \right|$$

$$d = \left| \frac{-3 \times 1 - 3 \times 0 + 3 \times (-2)}{3\sqrt{2}} \right|$$

$$d = \left|\frac{-3-6}{3\sqrt{2}}\right|$$

$$d = \left|\frac{-3}{\sqrt{2}}\right|$$

$$d = \frac{3}{\sqrt{2}}$$

$$d = \frac{3\sqrt{2}}{2}$$

15. Find the shortest distance between the lines:

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

Sol.

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

Direction ratios of the above lines will be:

$a_1 = 7,\ b_1 = -6,\ c_1 = 1$

$a_2 = 1,\ b_2 = -2,\ c_2 = 1$

$x_1 = -1,\ y_1 = -1,\ z_1 = -1$

$x_2 = 3,\ y_2 = 5,\ z_2 = 7$

Shortest distance between two lines,

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 4(-6+2) - 6(7-1) + 8(-14+6)$$

$$= -16 - 36 - 64$$

$$= -116$$

$$\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}$$

$$= \sqrt{16+36+64}$$

$$= 2\sqrt{29}$$

Now,

$$d = \frac{-116}{2\sqrt{29}} = -2\sqrt{29}$$

16. Find the shortest distance between the lines whose vector equations are:

$$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} - 3\hat{j} + 2\hat{k}\right) \text{ and}$$

$$\vec{r} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) + \mu\left(2\hat{i} + 3\hat{j} + \hat{k}\right)$$

Sol.

$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$ and

$\vec{r} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) + \mu\left(2\hat{i} + 3\hat{j} + \hat{k}\right)$

$\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$

Shortest distance between the above lines, $d = \left|\dfrac{\left(\vec{b_1} \times \vec{b_2}\right).\left(\vec{a_2} - \vec{a_1}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|}\right|$

$\vec{a_2} - \vec{a_1} = 4\hat{i} + 5\hat{j} + 6\hat{k} - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$

$= 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$= -9\hat{i} + 3\hat{j} + 9\hat{k}$

$\left|\vec{b_1} \times \vec{b_2}\right| = \sqrt{(-9)^2 + 3^2 + 9^2} = 3\sqrt{19}$

Now,

$$d = \left|\frac{\left(\vec{b_1} \times \vec{b_2}\right).\left(\vec{a_2} - \vec{a_1}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|}\right|$$

$$d = \left| \frac{\left(-9\hat{i} + 3\hat{j} + 9\hat{k}\right).\left(3\hat{i} + 3\hat{j} + 3\hat{k}\right)}{3\sqrt{19}} \right|$$

$$d = \left| \frac{-9 \times 3 + 3 \times 3 + 9 \times 3}{3\sqrt{19}} \right|$$

$$d = \left| \frac{-27 + 9 + 27}{3\sqrt{19}} \right|$$

$$d = \frac{3}{\sqrt{19}}$$

17. Find the shortest distance between the lines whose vector equations are:

$$\vec{\mathbf{r}} = (\mathbf{1 - t})\hat{\mathbf{i}} + (\mathbf{t - 2})\hat{\mathbf{j}} + (\mathbf{3 - 2t})\hat{\mathbf{k}} \text{ and}$$

$$\vec{\mathbf{r}} = (\mathbf{s + 1})\hat{\mathbf{i}} + (\mathbf{2s - 1})\hat{\mathbf{j}} - (\mathbf{2s + 1})\hat{\mathbf{k}}$$

Sol.

$$\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$$

$$\vec{r} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) + t\left(-\hat{i} + \hat{j} - 2\hat{k}\right)$$

$$\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$$

$$\vec{r} = \left(\hat{i} - \hat{j} + \hat{k}\right) + s\left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$$

$$\overrightarrow{a_1} = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{b_1} = -\hat{i} + \hat{j} - 2\hat{k}$$

$$\overrightarrow{a_2} = \hat{i} - \hat{j} + \hat{k}$$

$$\overrightarrow{b_2} = \hat{i} + 2\hat{j} - 2\hat{k}$$

Shortest distance between the above lines, $d = \left|\frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right|$

$$\vec{a_2} - \vec{a_1} = \hat{i} - \hat{j} + \hat{k} - (\hat{i} - 2\hat{j} + 3\hat{k})$$

$$= \hat{j} - 4\hat{k}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$$

$$d = \left|\frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right|$$

$$d = \left|\frac{(2\hat{i} - 4\hat{j} - 3\hat{k}).(\hat{j} - 4\hat{k})}{\sqrt{29}}\right|$$

$$d = \left|\frac{2 \times 0 - 4 \times 1 - 3 \times (-4)}{\sqrt{29}}\right|$$

$$d = \left|\frac{-4 + 12}{\sqrt{29}}\right|$$

$$d = \frac{8}{\sqrt{29}}$$

Exersise-11.3

1. In each of the following cases; determine the direction cosines of the normal to the plane and the distance from the origin.

(a) z = 2

(b) x + y + z = 1

(c) 2x+3y – z = 5

(d) 5y + 8 = O

Sol.

(a) Given, z = 2

The direction ratios of normal are = 0, 0, and 1

The equation of the plane is:

0x + 0y + z = 2

We know that,

In the form lx + my + nz = d, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Therefore, the direction cosines are = 0, 0, 1 & d = 2 units

(b) x + y + z = 1

The direction ratios of normal are = 1, 1, and 1

$$\Rightarrow \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

By dividing the equation with $\sqrt{3}$, equation becomes as:

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

In the form lx + my + nz = d, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Thus, the direction cosines of the normal are = $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

The distance of normal from the origin is = $\frac{1}{\sqrt{3}}$

(c) $2x + 3y - z = 5$

The direction ratios of normal are = 2, 3, and −1

$$\Rightarrow \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$$

By dividing the equation with $\sqrt{14}$, equation becomes as:

$$\frac{2x}{\sqrt{14}} + \frac{3y}{\sqrt{14}} - \frac{z}{\sqrt{14}} = \frac{5}{\sqrt{14}}$$

In the form lx + my + nz = d, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Thus, the direction cosines of the normal are = $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}$

The distance of normal from the origin is = $\frac{5}{\sqrt{14}}$

(d) $5y + 8 = 0$

Thus equation is:

$\Rightarrow 0x - 5y + 0z = 8$

The direction ratios of normal are = 0, −5, and 0

$$\Rightarrow \sqrt{0^2 + (-5)^2 + 0^2} = 5$$

By dividing the equation with 5, equation becomes as:

$$y = -\frac{8}{5}$$

$$-y = \frac{8}{5}$$

In the form lx + my + nz = d, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Thus, the direction cosines of the normal are = $0, -1, 0$

The distance of normal from the origin is = $\frac{8}{5}$

2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.

Sol.

$$\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$$

$$\left|\vec{n}\right| = \sqrt{3^2 + 5^2 + (-6)^2} = \sqrt{70}$$

Thus,

$$\hat{n} = \frac{\vec{n}}{\left|\vec{n}\right|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

We know that, the equation of the plane with position vector is:

$$\vec{r}.\hat{n} = d$$

$$\vec{r}.\left(\frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}\right) = 7$$

This is the vector equation of the plane.

3. Find the Cartesian equation of the following planes:

(a) $\vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=2$

(b) $\vec{r}.\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$

(c) $\vec{r}.\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$

Sol.

(a) Equation of the plane is:

$$\vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=2$$

For any arbitrary point P (x, y, z) on the plane, position vector is:

$$\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$$

Thus, equation of plane becomes as:

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right).\left(\hat{i}+\hat{j}-\hat{k}\right)=2$$

$$x+y-z=2$$

This is the Cartesian equation of the plane.

(b) Equation of plane is:

$$\vec{r}.\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$$

For any arbitrary point P (x, y, z) on the plane, position vector is:

$$x\hat{i}+y\hat{j}-z\hat{k}$$

Thus, equation becomes as:

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right).\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=2$$

$$2x+3y-4z=1$$

This is the Cartesian equation of the plane.

(c) Equation of plane is:

$$\vec{r}.\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$$

For any arbitrary point P (x, y, z) on the plane, position vector is:

$$\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$$

Thus, the equation becomes as:

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right).\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$$

$$(s-2t)x+(3-t)y+(2s+t)z=15$$

This is the Cartesian equation of the given plane.

4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(a) 2x + 3y + 4z – 12 = 0
(b) 3y + 4z – 6 = 0
(c) x + y + z = 1
(d) 5y + 8 = 0

Sol.

(a)Let the coordinates of the foot of perpendicular P from the origin to the plane is:

$P = (x_1, y_1, z_1)$

$2x + 3y + 4z - 12 = 0$

$2x + 3y + 4z = 12$

The direction ratios of normal are = 2, 3, 4

$$\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

By dividing the equation by $\sqrt{29}$, we have

$$\frac{2}{\sqrt{29}}x + \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{12}{\sqrt{29}}$$

In the form lx + my + nz = d, l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are:

= (ld, md, nd)

$$= \left(\frac{2}{\sqrt{29}}.\frac{12}{\sqrt{29}} + \frac{3}{\sqrt{29}}.\frac{12}{\sqrt{29}} + \frac{4}{\sqrt{29}}.\frac{12}{\sqrt{29}}\right)$$

$$= \left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$$

(b) Let the coordinates of the foot of perpendicular P from the origin to the plane is:

$P = (x_1, y_1, z_1)$

$3y + 4z - 6 = 0$

$0x + 3y + 4z = 6$

The direction ratios of normal are = 0, 3, 4

$$\sqrt{0^2 + 3^2 + 4^2} = 5$$

By dividing the equation by 5, we have

$$0x + \frac{3}{5}y + \frac{4}{5}z = \frac{6}{5}$$

In the form lx + my + nz = d, l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are:

= (ld, md, nd)

$$= \left(0, \frac{3}{5}.\frac{6}{5}, \frac{4}{5}.\frac{6}{5}\right)$$

$$= \left(0, \frac{18}{25}, \frac{24}{25}\right)$$

(C) Let the coordinates of the foot of perpendicular P from the origin to the plane is:

$P = (x_1, y_1, z_1)$

$x + y + z = 1$

The direction ratios of normal are = 1, 1, 1

$$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

By dividing the equation by $\sqrt{3}$, we have

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$$

In the form $lx + my + nz = d$, l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are:

= (ld, md, nd)

$$= \left(\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

(d) Let the coordinates of the foot of perpendicular P from the origin to the plane is:

$P = (x_1, y_1, z_1)$

5y + 8 = 0

0x + 5y + 0z = 8

The direction ratios of normal are = 0, 5, 0

$$\sqrt{0^2 + 5^2 + 0^2} = 5$$

By dividing the equation by 5, we have

$$0x - y + 0z = \frac{8}{5}$$

In the form lx + my + nz = d, l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are:

= (ld, md, nd)

$$= \left(0, (-1).\frac{8}{5}, 0\right)$$

$$= \left(0, \frac{-8}{5}, 0\right)$$

5. Find the vector and Cartesian equation of the planes

(a) That passes through the point (1, 0, −2) and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.

(b) That passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

Sol.

(a) The position vector of point (1, 0, −2) is:

$$\vec{a} = \hat{i} - 2\hat{k}$$

The normal vector perpendicular to the plane, $\vec{n} = \hat{i} + \hat{j} - \hat{k}$

The vector equation of the plane is:

$$\left(\vec{r} - \vec{a}\right).\vec{n} = 0$$

$$\left[\vec{r} - \left(\hat{i} - 2\hat{k}\right)\right].\left(\hat{i} + \hat{j} - \hat{k}\right) = 0$$

Here, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Thus, vector equation of plane becomes as:

$$\left[\left(x\hat{i} + y\hat{j} + z\hat{k}\right) - \left(\hat{i} - 2\hat{k}\right)\right].\left(\hat{i} + \hat{j} - \hat{k}\right) = 0$$

$$\left[(x-1)\hat{i} + y\hat{j} - (z+2)\hat{k}\right].\left(\hat{i} + \hat{j} - \hat{k}\right) = 0$$

$$(x-1) + y - (z+2) = 0$$

$$x + y - z = 3$$

This is the Cartesian equation of the required plane.

(b) The position vector of point (1, 4, 6) is:

$$\vec{a} = \hat{i} + 4\hat{j} + 6\hat{k}$$

The normal vector perpendicular to the plane, $\vec{n} = \hat{i} - 2\hat{j} + \hat{k}$

The vector equation of the plane is:

$$\left(\vec{r} - \vec{a}\right).\vec{n} = 0$$

$$\left[\vec{r} - \left(\hat{i} + 4\hat{j} + 6\hat{k}\right)\right].\left(\hat{i} - 2\hat{j} + \hat{k}\right) = 0$$

Here, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Thus, vector equation of plane becomes as:

$$\left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$$

$$\left[(x-1)\hat{i}+(y-4)\hat{j}+(z-6)\hat{k}\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$$

$$(x-1)+(y-4)+(z-6)=0$$

$$x-2y+z=-1$$

This is the Cartesian equation of the required plane.

6. Find the equations of the planes that pass through three points.

(a) (1, 1, −1), (6, 4, −5), (−4, −2, 3)

(b) (1, 1, 0), (1, 2, 1), (−2, 2, −1)

Sol.

(a) A = (1, 1, −1)

B (6, 4, −5)

C (−4, −2, 3)

Now,

$$\begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{vmatrix}$$

$$=1(12-10)-1(18-20)-1(-12+16)$$

$$=2+2-4$$

$$=0$$

A, B, C are collinear points. Thus, there will be infinite number of planes passing through the given points.

(b) A = (1, 1, 0)

B (1, 2, 1)

C (−2, 2, −1)

Now,

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix}$$

$$= 1(-2-2) - 1(-1+2) + 0$$

$$= -4 - 1$$

$$= -5 \neq 0$$

Thus, a plane passes through the points A, B, and C.

The equation of the plane through the points is:

$$\begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$$(x-1)(-1-1) - (y-1)(0+3) + z(0+3) = 0$$

$$-2x + 2 - 3y + 3 + 3z = 0$$

$$-2x - 3y + 3z + 5 = 0$$

$$2x + 3y - 3z = 5$$

This is the Cartesian equation of the required plane.

7. Find the intercepts cut off by the plane 2x + y – z = 5

Sol.

$2x + y - z = 5$

Divide the equation by 5

$$\frac{2}{5}x + \frac{1}{5}y - \frac{1}{5}z = 1$$

$$\frac{1}{\frac{5}{2}}x + \frac{1}{5}y - \frac{1}{5}z = 1$$

Thus, the intercepts cut off by the plane are = $\left(\frac{5}{2}, 5, -5\right)$

8. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

Sol.

The equation of the plane ZOX is: $y = 0$

A plane parallel to the given plane is: $y = a$

We have, $y = 3$

We get, $a = 3$

Hence, the equation of the required plane is $y = 3$

9. Find the equation of the plane through the intersection of the planes $3x – y + 2z – 4 = 0$ and $x + y + z − 2 = 0$ and the point (2, 2, 1)

Sol.

Equations of planes are:

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$

The equation of plane through the intersection of the above planes is:

$$(3x - y + 2z - 4) + \alpha(x + y + z - 2) = 0$$

Point = (2, 2, 1)

$$(3\times 2-2+2\times 1-4)+\alpha\,(2+2+1-2)=0$$

$$2+3\alpha=0$$

$$\alpha=-\frac{2}{3}$$

The equation of required plane becomes as:

$$(3x-y+2z-4)-\frac{2}{3}(x+y+z-2)=0$$

$$3(3x-y+2z-4)-2(x+y+z-2)=0$$

$$9x-3y+6z-12-2x-2y-2z+4=0$$

$$7x-5y+4z-8=0$$

10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=7$, $\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$ and through the point (2, 1, 3)

Sol.

Equations of planes are:

$$\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=7 \quad \& \quad \vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$$

$$\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7=0 \quad \& \quad \vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9=0$$

The equation of a plane passing through the intersection of the given planes is:

$$\left[\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7\right]+\lambda\left[\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9\right]=0$$

$$\vec{r}\left[.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7+\lambda\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9\right]=0$$

$$\vec{r}\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}\right]=9\lambda+7$$

Point = (2, 1, 3)

Position vector of plane passing through the give point is:

$$\vec{r} = 2\hat{i} + \hat{j} + 3\hat{k}$$

Now, the equation of desired plane becomes as:

$$\left(2\hat{i} + \hat{j} + 3\hat{k}\right)\left[(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}\right] = 9\lambda + 7$$

$$(2 + 2\lambda)2 + (2 + 5\lambda) + (3\lambda - 3)3 = 9\lambda + 7$$

$$4 + 4\lambda + 2 + 5\lambda + 9\lambda - 9 = 9\lambda + 7$$

$$9\lambda = 10$$

$$\lambda = \frac{10}{9}$$

Now,

$$\vec{r}\left[(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}\right] = 9\lambda + 7$$

$$\vec{r}\left[\left(2 + 2 \times \frac{10}{9}\right)\hat{i} + \left(2 + 5 \times \frac{10}{9}\right)\hat{j} + \left(3 \times \frac{10}{9} - 3\right)\hat{k}\right] = 9\lambda + 7$$

$$\vec{r}\left[\frac{38}{9}\hat{i} + \frac{68}{9}\hat{j} + \frac{3}{9}\hat{k}\right] = 10 + 7$$

$$\vec{r}\left(38\hat{i} + 68\hat{j} + 3\hat{k}\right) = 153$$

This is the vector equation of the required plane.

11. Find the equation of the plane through the line of intersection of the planes x + y + z = 1 and 2x + 3y + 4z = 5 which is perpendicular to the plane x – y + z = 0

Sol.

Equations of planes are:

x + y + z = 1 and 2x + 3y + 4z = 5

The equation of the plane passing through the intersection of the given planes is:

$$(x+y+z-1)+\lambda(2x+3y+4z-5)=0$$
$$(2\lambda+1)x+(3\lambda+1)y+(4\lambda+1)z-(5\lambda+1)=0$$

The direction ratios of this plane are:

$a_1 = (2\lambda + 1)$

$b_1 = (3\lambda + 1)$

$c_1 = (4\lambda + 1)$

The given plane x – y + z = 0 is perpendicular to the equation of desired plane

Direction ratios of plane, x – y + z = 0 are:

$a_2 = 1$

$b_2 = -1$

$c_2 = 1$

Both planes are perpendicular. Thus,

$$a_1a_2+b_1b_2+c_1c_2=0$$
$$(2\lambda+1)-(3\lambda+1)+(4\lambda+1)=0$$
$$3\lambda+1=0$$
$$\lambda=-\frac{1}{3}$$

Equation of desired plane becomes as:

$$\left(2\times\left(\frac{-1}{3}\right)+1\right)x+\left(3\times\left(\frac{-1}{3}\right)+1\right)y+\left(4\times\left(\frac{-1}{3}\right)+1\right)z-\left(5\times\left(\frac{-1}{3}\right)+1\right)=0$$
$$\frac{1}{3}x-\frac{1}{3}z+\frac{2}{3}=0$$
$$x-z+2=0$$

This is the required equation of the plane.

12. Find the angle between the planes whose vector equations are
$\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ **and** $\vec{r}.\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$

Sol.

The equations of the given planes are:

$$\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$$

$$\vec{r}.\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$$

Now,

$$\vec{n_1}=2\hat{i}+2\hat{j}-3\hat{k}$$
$$\vec{n_2}=3\hat{i}-3\hat{j}+5\hat{k}$$

Angle between the given pair of lines, $\cos\theta=\left|\dfrac{\vec{n_1}.\vec{n_2}}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|}\right|$

$$\begin{aligned}\vec{n_1}.\vec{n_2}&=\left(2\hat{i}+2\hat{j}-3\hat{k}\right)\left(3\hat{i}-3\hat{j}+5\hat{k}\right)\\&=2\times3+2\times(-3)-3\times5\\&=6-6-15\\&=-15\end{aligned}$$

$$\left|\vec{n_1}\right|=\sqrt{2^2+2^2+(-3)^2}=\sqrt{17}$$
$$\left|\vec{n_2}\right|=\sqrt{3^2+(-3)^2+5^2}=\sqrt{43}$$

$$\cos\theta=\left|\frac{\vec{n_1}.\vec{n_2}}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|}\right|$$

$$\cos\theta=\left|\frac{-15}{\sqrt{17}.\sqrt{43}}\right|$$

$$\cos\theta=\frac{15}{\sqrt{731}}$$

13. **In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them:**

a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Sol.

a) The equations of planes are:

$7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

$a_1 = 7, b_1 = 5, c_1 = 6$

$a_2 = 3, b_2 = -1, c_2 = -10$

Now,

$$\frac{a_1}{a_2} = \frac{7}{3}, \quad \frac{b_1}{b_2} = \frac{-5}{1}, \quad \frac{c_1}{c_2} = \frac{-3}{5}$$

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Thus, these lines are not parallel

$a_1a_2 + b_1b_2 + c_1c_2$

$= 7 \times 3 + 5 \times (-1) + 6 \times (-10)$

$= 21 - 5 - 60$

$= -44 \neq 0$

hence, these lines are not $\perp$

Now,

$$\cos Q = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

$$\cos Q = \left| \frac{-44}{\sqrt{7^2 + 5^2 + 6^2}.\sqrt{3^2 + (-1)^2 + (-10)^2}} \right|$$

$$Q = \cos^{-1}\left| \frac{-44}{\sqrt{110}.\sqrt{110}} \right|$$

$$Q = \cos^{-1}\left(\frac{2}{5} \right)$$

b) The equations of planes are:

$2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

$a_1 = 2, b_1 = 1, c_1 = 3$

$a_2 = 1, b_2 = -2, c_2 = 0$

Now,

$a_1a_2 + b_1b_2 + c_1c_2$

$= 2\times 1 + 1\times(-2) + 3\times(0)$

$= 2 - 2 - 0$

$= 0$

Hence, these lines are perpendicular to each other.

c) The equations of planes are:

$2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

$a_1 = 2, b_1 = -2, c_1 = 4$

$a_2 = 3, b_2 = -3, c_2 = 6$

Now,

$$\frac{a_1}{a_2} = \frac{2}{3}, \quad \frac{b_1}{b_2} = \frac{2}{3}, \quad \frac{c_1}{c_2} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, these lines are parallel to each other.

d) The equations of planes are:

$2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

$a_1 = 2, b_1 = -1, c_1 = 3$

$a_2 = 2, b_2 = -1, c_2 = 3$

Now,

$$\frac{a_1}{a_2} = \frac{2}{2}, \quad \frac{b_1}{b_2} = \frac{-1}{-1}, \quad \frac{c_1}{c_2} = \frac{3}{3}$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, these lines are parallel to each other.

e) The equations of planes are:

$4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

$a_1 = 4, b_1 = 8, c_1 = 1$

$a_2 = 0, b_2 = 1, c_2 = 1$

Now,

$$\frac{a_1}{a_2} = \frac{4}{0}, \quad \frac{b_1}{b_2} = \frac{8}{1}, \quad \frac{c_1}{c_2} = \frac{1}{1}$$

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Thus, these lines are not parallel

$a_1 a_2 + b_1 b_2 + c_1 c_2$

$= 4 \times 0 + 8 \times 1 + 1 \times 1$

$= 8 + 1$

$= 9$

Hence, these lines are not perpendicular as well

$Now,$

$$\cos Q = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

$$\cos Q = \left| \frac{9}{\sqrt{4^2 + 8^2 + 1^2}.\sqrt{0^2 + 1^2 + 1^2}} \right|$$

$$Q = \cos^{-1}\left| \frac{9}{9\sqrt{2}} \right|$$

$$Q = \cos^{-1}\left(\frac{1}{\sqrt{2}} \right)$$

$$Q = 45^\circ$$

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

Point Plane

(a) (0, 0, 0) 3x - 4y + 12z = 3

(b) (3, −2, 1) 2x - y + 2z +3 = 0

(c) (2, 3, −5) x + 2y - 2z = 9

(d) (−6, 0, 0) 2x - 3y + 6z -2 = 0

Sol.

1.

$$P = (0, 0, 0)$$

Equation of plane is:

$$3x - 4y + 12z = 3$$

Distance between a point $P(x_1, y_1, z_1)$ & the plane $ax + by + cz = d$ is:

$$d = \left|\frac{ax_1 + by_1 + cz_1 - D}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$d = \left|\frac{3 \times 0 - 4 \times 0 + 12 \times 0 - 3}{\sqrt{3^2 + (-4)^2 + (12)^2}}\right|$$

$$d = \frac{3}{13}$$

2.

$P = (3, -2, 1)$

Equation of plane is:

$2x - y + 2z + 3 = 0$

Distance between a point $P(x_1, y_1, z_1)$ & the plane $ax + by + cz = d$ is:

$$d = \left|\frac{ax_1 + by_1 + cz_1 - D}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$d = \left|\frac{2 \times 3 + (-2) \times (-1) + 2 \times 1 + 3}{\sqrt{2^2 + (-1)^2 + (2)^2}}\right|$$

$$d = \left|\frac{6 + 2 + 2 + 3}{\sqrt{9}}\right|$$

$$d = \frac{13}{3}$$

3.

$P = (2, 3, -5)$

Equation of plane is:

$x + 2y - 2z = 9$

Distance between a point $P(x_1, y_1, z_1)$ & the plane $ax + by + cz = d$ is:

$$d = \left|\frac{ax_1 + by_1 + cz_1 - D}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$d = \left|\frac{2 \times 1 + 3 \times 2 + (-5) \times (-2) - 9}{\sqrt{1^2 + 2^2 + (-2)^2}}\right|$$

$$d = \left|\frac{2 + 6 + 10 - 9}{\sqrt{9}}\right|$$

$$d = \frac{9}{3} = 3$$

4.

$P = (-6, 0, 0)$

Equation of plane is:

$2x - 3y + 6z - 2 = 0$

Distance between a point $P(x_1, y_1, z_1)$ & the plane $ax + by + cz = d$ is:

$$d = \left|\frac{ax_1 + by_1 + cz_1 - D}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$d = \left|\frac{2 \times (-6) - 3 \times 0 + 6 \times 0 - 2}{\sqrt{2^2 + (-3)^2 + 6^2}}\right|$$

$$d = \left|\frac{-12 - 2}{\sqrt{49}}\right|$$

$$d = \frac{14}{7} = 2$$